

$\ell^p(\mathbb{Z}^d)$ -ESTIMATES FOR DISCRETE OPERATORS OF RADON TYPE: VARIATIONAL ESTIMATES

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ABSTRACT. We prove $\ell^p(\mathbb{Z}^d)$ bounds for $p \in (1, \infty)$, of r -variations $r \in (2, \infty)$, for discrete averaging operators and truncated singular integrals of Radon type. We shall present a new powerful method which allows us to deal with these operators in a unified way and obtain the range of parameters of p and r which coincide with the ranges of their continuous counterparts.

1. INTRODUCTION

In this paper we will be concerned with estimates of r -variations for discrete operators of averaging and singular Radon type, and their application to ergodic theory. The r -variational estimates for the continuous versions of these operators will be treated as well.

Let

$$\mathcal{P} = (\mathcal{P}_1, \dots, \mathcal{P}_{d_0}) : \mathbb{Z}^k \rightarrow \mathbb{Z}^{d_0}$$

be a polynomial mapping where for each $j \in \{1, \dots, d_0\}$ the function $\mathcal{P}_j : \mathbb{Z}^k \rightarrow \mathbb{Z}$ is an integer valued polynomial of k variables with $\mathcal{P}_j(0) = 0$. Define, for a finitely supported function $f : \mathbb{Z}^{d_0} \rightarrow \mathbb{C}$, the Radon averages

$$(1.1) \quad M_N^{\mathcal{P}} f(x) = |\mathbb{B}_N|^{-1} \sum_{y \in \mathbb{B}_N} f(x - \mathcal{P}(y))$$

where $\mathbb{B}_t = \{x \in \mathbb{Z}^k : |x| \leq t\}$ and $t > 0$. We will be also interested in discrete truncated singular integrals.

Assume that $K \in C^1(\mathbb{R}^k \setminus \{0\})$ is a Calderón–Zygmund kernel satisfying the differential inequality

$$|y|^k |K(y)| + |y|^{k+1} |\nabla K(y)| \leq 1$$

for all $y \in \mathbb{R}^k$ with $|y| \geq 1$. We also impose the following cancellation condition

$$(1.2) \quad \int_{B_{\lambda_2} \setminus B_{\lambda_1}} K(y) \, dy = 0$$

for every $0 < \lambda_1 \leq \lambda_2$ where B_λ is the Euclidean ball in \mathbb{R}^k centered at the origin with radius $\lambda > 0$. Define, for a finitely supported function $f : \mathbb{Z}^{d_0} \rightarrow \mathbb{C}$, the truncated singular Radon transforms

$$(1.3) \quad T_N^{\mathcal{P}} f(x) = \sum_{y \in \mathbb{B}_N \setminus \{0\}} f(x - \mathcal{P}(y)) K(y).$$

The basic aim of this paper is to strengthen the $\ell^p(\mathbb{Z}^{d_0})$ boundedness, $p \in (1, \infty)$, of maximal functions corresponding to operators (1.1) and (1.3), which have been recently proven in [15], and provide sharp r -variational bounds in the full range of exponents.

Recall that for any $r \in [1, \infty)$ the r -variational seminorm V_r of a sequence $(a_n(x) : n \in \mathbb{N})$ of complex-valued functions is defined by

$$V_r(a_n(x) : n \in \mathbb{N}) = \sup_{\substack{k_0 < \dots < k_J \\ k_j \in \mathbb{N}}} \left(\sum_{j=1}^J |a_{k_j}(x) - a_{k_{j-1}}(x)|^r \right)^{1/r}.$$

The main results of this article are the following theorems.

Theorem A. *For every $p \in (1, \infty)$ and $r \in (2, \infty)$ there is $C_{p,r} > 0$ such that for all $f \in \ell^p(\mathbb{Z}^{d_0})$*

$$(1.4) \quad \|V_r(M_N^{\mathcal{P}} f : N \in \mathbb{N})\|_{\ell^p} \leq C_{p,r} \|f\|_{\ell^p}.$$

Moreover, the constant $C_{p,r} \leq C_p \frac{r}{r-2}$ for some $C_p > 0$ which is independent of the coefficients of the polynomial mapping \mathcal{P} .

We also obtain the corresponding theorem for the truncated singular Radon transforms.

Theorem B. *For every $p \in (1, \infty)$ and $r \in (2, \infty)$ there is $C_{p,r} > 0$ such that for all $f \in \ell^p(\mathbb{Z}^{d_0})$*

$$(1.5) \quad \|V_r(T_N^{\mathcal{P}} f : N \in \mathbb{N})\|_{\ell^p} \leq C_{p,r} \|f\|_{\ell^p}.$$

Moreover, the constant $C_{p,r} \leq C_p \frac{r}{r-2}$ for some $C_p > 0$ which is independent of the coefficients of the polynomial mapping \mathcal{P} .

Theorem A and Theorem B have ergodic theoretical interpretations. More precisely, let (X, \mathcal{B}, μ) be a σ -finite measure space with a family of invertible commuting and measure preserving transformations S_1, S_2, \dots, S_{d_0} . Let

$$(1.6) \quad \mathcal{A}_N^{\mathcal{P}} f(x) = N^{-k} \sum_{y \in \mathbb{B}_N} f(S_1^{\mathcal{P}_1(y)} S_2^{\mathcal{P}_2(y)} \dots S_{d_0}^{\mathcal{P}_{d_0}(y)} x)$$

and

$$(1.7) \quad \mathcal{H}_N^{\mathcal{P}} f(x) = \sum_{y \in \mathbb{B}_N \setminus \{0\}} f(S_1^{\mathcal{P}_1(y)} S_2^{\mathcal{P}_2(y)} \dots S_{d_0}^{\mathcal{P}_{d_0}(y)} x) K(y).$$

Specifying a suitable measure space (X, \mathcal{B}, μ) and a family of measure preserving transformations we immediately see that $\mathcal{A}_N^{\mathcal{P}}$ and $\mathcal{H}_N^{\mathcal{P}}$ coincide with $M_N^{\mathcal{P}}$ and $T_N^{\mathcal{P}}$ respectively. Indeed, it suffices to take $X = \mathbb{Z}^{d_0}$, $\mathcal{B} = \mathbf{P}(\mathbb{Z}^{d_0})$ σ -algebra of all subsets of \mathbb{Z}^{d_0} , $\mu = |\cdot|$ to be the counting measure on \mathbb{Z}^{d_0} and $S_j^y : \mathbb{Z}^{d_0} \rightarrow \mathbb{Z}^{d_0}$ the shift operator acting of j -th coordinate, i.e. $S_j^y(x_1, \dots, x_{d_0}) = (x_1, \dots, x_j - y, \dots, x_{d_0})$ for all $j = 1, 2, \dots, d_0$ and $y \in \mathbb{Z}$.

Theorem C. *Let $\mathcal{S}_N^{\mathcal{P}}$ be the operator given either by (1.6) or by (1.7) and assume that $p \in (1, \infty)$ and $r \in (2, \infty)$. Then there is $C_{p,r} > 0$ such that for all $f \in L^p(X, \mu)$*

$$(1.8) \quad \|V_r(\mathcal{S}_N^{\mathcal{P}} f : N \in \mathbb{N})\|_{L^p} \leq C_{p,r} \|f\|_{L^p}.$$

In particular, (1.8) implies that for every $f \in L^p(X, \mu)$ there exists $f^ \in L^p(X, \mu)$ such that*

$$\lim_{N \rightarrow \infty} \mathcal{S}_N^{\mathcal{P}} f(x) = f^*(x)$$

μ -almost everywhere on X .

The estimate (1.8) from Theorem C can be deduced from inequality (1.4) or (1.5) by appealing to the Calderón transference principle. Furthermore, Theorem C with $\mathcal{S}_N^{\mathcal{P}} = \mathcal{H}_N^{\mathcal{P}}$ can be thought as an extension of Cotlar's ergodic theorem (see [4]), which states that for every σ -finite measure space (X, \mathcal{B}, μ) with an invertible and a measure preserving transformation S the limit

$$\lim_{N \rightarrow \infty} \sum_{0 < |n| \leq N} \frac{f(S^n x)}{n}$$

exists μ -almost everywhere on X for every $f \in L^p(X, \mu)$ with $p \in (1, \infty)$.

The classical strategy for handling pointwise convergence problems requires $L^p(X, \mu)$ boundedness for the corresponding maximal function, reducing the matters to proving pointwise convergence for a dense class of $L^p(X, \mu)$ functions. However, establishing pointwise convergence on a dense class can be a quite challenging problem. This is the case for Bourgain's averaging operator along the squares. Fortunately in [1], he was able to circumvent this issue for the operators $\mathcal{A}_N^{\mathcal{P}}$, in the one dimensional case $k = d_0 = 1$, by controlling their oscillation seminorm. Given a lacunary sequence $(n_j : j \in \mathbb{N})$, the oscillation seminorm for a sequence $(a_n : n \in \mathbb{N})$ of complex numbers is defined by

$$O_J(a_n : n \in \mathbb{N}) = \left(\sum_{j=1}^J \sup_{n_j < n \leq n_{j+1}} |a_n - a_{n_j}|^2 \right)^{1/2}.$$

Bourgain deduced pointwise convergence on $L^2(X, \mu)$ for the operators $\mathcal{A}_N^{\mathcal{P}}$ by proving, roughly, that there are constants $C > 0$ and $c < 1/2$ such that for all $J \in \mathbb{N}$

$$\|O_J(\mathcal{A}_N^{\mathcal{P}} f : N \in \mathbb{N})\|_{L^2} \leq C J^c \|f\|_{L^2}.$$

Variational estimates have been the subject of many papers, see [9, 10, 11, 16, 22] and the references therein. Our motivation to study r -variational seminorms is threefold. Firstly, for any sequence of functions $(a_n(x) : n \in \mathbb{N})$, if for some $1 \leq r < \infty$

$$V_r(a_n(x) : n \in \mathbb{N}) < \infty$$

then the limit $\lim_{n \rightarrow \infty} a_n(x)$ exists. Secondly, V_r 's control the supremum norm. Indeed, for any $n_0 \in \mathbb{N}$ we have the pointwise estimate

$$\sup_{n \in \mathbb{N}} |a_n(x)| \leq |a_{n_0}(x)| + 2V_r(a_n(x) : n \in \mathbb{N}).$$

Furthermore, for any $2 \leq r < \infty$ by Hölder's inequality we have

$$O_J(a_n(x) : n \in \mathbb{N}) \leq J^{1/2-1/r} V_r(a_n(x) : n \in \mathbb{N}).$$

The variational estimates for Bourgain's averaging operator (1.1) with $k = d_0 = 1$, have recently been extensively studied, while only partial results were obtained. Namely, Krause [11] showed inequality (1.4) for $p \in (1, \infty)$ and $r > \max\{p, p'\}$. Zorin-Kranich [22] showed (1.4) with $r \in (2, \infty)$ and $p \in (1, \infty)$ in some vicinity of 2, i.e. $|1/p - 1/2| < 1/(2(d' + 1))$, where d' is the degree of the polynomial. Their proofs were based on variational estimates of the famous Bourgain's logarithmic lemma provided by Nazarov, Oberlin and Thiele in [17], see also [12] for some improvements. That was the main building block in their arguments. Although the logarithmic lemma gives very nice $\ell^2(\mathbb{Z})$ results, it is generally very inefficient for $\ell^p(\mathbb{Z})$. The reason, loosely speaking, is that it produces for $p \neq 2$ a polynomial growth in norm unlike the acceptable logarithmic growth which one has for $p = 2$. Therefore in this paper we introduce a different flexible approach based on a direct analysis of the multipliers associated with operators (1.1) and (1.3), and instead of Bourgain's logarithmic lemma we will apply a simple numerical inequality, see Lemma 2.1, which turns out to be a more appropriate tool in these problems with arithmetic flavor. This lemma is a variant of the crucial Lemma 2.2 that we used in [15].

The proof of Theorem A and Theorem B, in view of inequality (2.5), will be based on separate estimates for long and short variational seminorms of the operators $M_N^{\mathcal{P}}$ and $T_N^{\mathcal{P}}$ respectively. We now describe the key points of our method in the case of averaging operator (1.1). Assume, for simplicity, that $k = 1$ and $\mathcal{P}(x) = (x^d, \dots, x)$ is a moment curve for some $d_0 = d \geq 2$. Let m_N be the multiplier associated with $M_N^{\mathcal{P}}$, i.e. $\mathcal{F}^{-1}(m_N \hat{f}) = M_N^{\mathcal{P}} f$.

The estimates of long variations will be very much in spirit of the estimates of maximal functions associated with $M_N^{\mathcal{P}}$ as we gave in [15]. For this purpose as in [15] we introduce an appropriate partition of unity which permits us to identify asymptotic or highly oscillatory behaviour of m_{2^n} , corresponding respectively to the "major" and "minor" arcs. This distinction is based on the ideas of the circle method of Hardy and Littlewood. More precisely, let η be a smooth cut-off function with a small support, fix $l \in \mathbb{N}$ and for each $n \in \mathbb{N}$ define projections

$$\Xi_n(\xi) = \sum_{a/q \in \mathcal{U}_{n^l}} \eta(\mathcal{E}_n^{-1}(\xi - a/q))$$

where \mathcal{E}_n is a diagonal $d \times d$ matrix with positive entries $(\varepsilon_j : 1 \leq j \leq d)$ such that $\varepsilon_j \leq e^{-n^{1/5}}$ and

$$\mathcal{U}_{n^l} = \{a/q \in \mathbb{T}^d \cap \mathbb{Q}^d : a = (a_1, \dots, a_d) \in \mathbb{N}_q^d \text{ and } \gcd(a_1, \dots, a_d, q) = 1 \text{ and } q \in P_{n^l}\}$$

for some family P_{n^l} such that $\mathbb{N}_{n^l} \subseteq P_{n^l} \subseteq \mathbb{N}_{e^{n^{1/10}}}$, we refer to the last subsection of Section 3 for more detailed definitions. The projections Ξ_n will be critical in the further analysis, since

$$V_r^L(M_N^{\mathcal{P}} f : N \in \mathbb{N}) \leq V_r(\mathcal{F}^{-1}(m_{2^n} \Xi_n \hat{f}) : n \in \mathbb{N}_0) + V_r(\mathcal{F}^{-1}(m_{2^n} (1 - \Xi_n) \hat{f}) : n \in \mathbb{N}_0)$$

where $m_{2^n}(\xi) \Xi_n(\xi)$ corresponds to the asymptotic behaviour of $m_{2^n}(\xi)$, whereas $m_{2^n}(\xi)(1 - \Xi_n(\xi))$ localizes the highly oscillatory part. For the last piece we can prove that

$$\|V_r(\mathcal{F}^{-1}(m_{2^n} (1 - \Xi_n) \hat{f}) : n \in \mathbb{N}_0)\|_{\ell^p} \leq C_{p,r} \|f\|_{\ell^p}.$$

These bounds can be deduced from a variant of Weyl's inequality with logarithmic decay, see Theorem 3.1 or [15], and the following inequality for $p \in (1, \infty)$, that goes back to ideas of Ionescu and Wainger,

$$(1.9) \quad \|\mathcal{F}^{-1}(\Xi_n \hat{f})\|_{\ell^p} \leq C_{l,p} \log(n+2) \|f\|_{\ell^p}$$

see Theorem 3.2 or [15] and [8] for more detailed expositions. For the proof of the estimate

$$\|V_r(\mathcal{F}^{-1}(m_{2^n} \Xi_n \hat{f}) : n \in \mathbb{N}_0)\|_{\ell^p} \leq C_{p,r} \|f\|_{\ell^p},$$

we show that

$$m_{2^n}(\xi) \Xi_n(\xi) \simeq \sum_{s \geq 0} m_{2^n}^s(\xi)$$

where

$$(1.10) \quad m_{2^n}^s(\xi) = \sum_{a/q \in \mathcal{U}_{(s+1)l} \setminus \mathcal{U}_{sl}} G(a/q) \Phi_{2^n}(\xi - a/q) \eta(\mathcal{E}_s^{-1}(\xi - a/q)),$$

with $G(a/q)$ being the Gaussian sum and Φ_{2^n} being the continuous version of m_{2^n} , see at the beginning of Section 4 for relevant definitions. Then the matters are reduced to proving that for each $s \geq 0$ we have

$$(1.11) \quad \|V_r(\mathcal{F}^{-1}(m_{2^n}^s \Xi_n \hat{f}) : n \in \mathbb{N}_0)\|_{\ell^p} \leq C_p(s+1)^{-2} \|f\|_{\ell^p}.$$

Firstly, we prove (1.11) for $p = 2$ with bound $C_r(s+1)^{-\delta l+1} \|f\|_{\ell^2}$, where $\delta > 0$ is an exponent from the bound for the Gaussian sums $|G(a/q)| \leq Cq^{-\delta}$, and $l \in \mathbb{N}$ an arbitrary integer. Secondly, for general $p \neq 2$ we obtain much worse bound $C_{l,p,r} s \log(s+2) \|f\|_{\ell^p}$. Now interpolating the last bound with much better for $p = 2$ we get the claim from (1.11), since $l \in \mathbb{N}$ can be arbitrarily large. To achieve both bounds we partition the V_r in (1.11) into two pieces $n \leq 2^{\kappa_s}$ and $n > 2^{\kappa_s}$ for some integer $1 < \kappa_s \leq Cs$. The case for large scales $n > 2^{\kappa_s}$ follows by invoking the transference principle which allows us to control discrete $\|\cdot\|_{\ell^p}$ norm of r -variations associated with multipliers from (1.10) by the continuous $\|\cdot\|_{L^p}$ norm of r -variations closely related with the multiplier Φ_{2^n} , which is *a priori* bounded on \mathbb{R}^d . This is the place, and only place, where we are restricted to $r \in (2, \infty)$, and then obtain the growth of the constant $C_{p,r} \leq C_p \frac{r}{r-2}$ in Theorem A. The reason lies in application Lépingle's inequality, (see Theorem A.5 and Theorem A.6 in the Appendix) to bound $L^p(\mathbb{R}^d)$ norm. The case of small scales $n \leq 2^{\kappa_s}$ has different nature and some new ideas came up. An invaluable tool which surmounted complications occurring in [11] and [22] is a simple numerical inequality from Lemma 2.1 yielding

$$(1.12) \quad V_r(\mathcal{F}^{-1}(m_{2^n}^s \hat{f}) : 0 \leq n \leq 2^{\kappa_s}) \leq \sqrt{2} \sum_{i=0}^{\kappa_s} \left(\sum_{j=0}^{2^{\kappa_s-i}-1} |\mathcal{F}^{-1}(m_{2^{(j+1)2^i}}^s \hat{f}) - \mathcal{F}^{-1}(m_{2^{j2^i}}^s \hat{f})|^2 \right)^{1/2}.$$

Applying now Theorem 3.2 (see also [15]) we shall show that $\ell^p(\mathbb{Z}^d)$ norm of the inner square function on the right-hand side in (1.12) is bounded by $C_p \log(s+2) \|f\|_{\ell^p}$ for each $0 \leq i \leq \kappa_s$. Consequently, we get the desired bound since there are $\kappa_s + 1$ elements. This illustrates roughly the scheme for long r -variations.

In order to attack short variations we will again exploit the partition of unity introduced above and obtain

$$(1.13) \quad V_r^S(M_N^P f : N \in \mathbb{N}) \leq \left(\sum_{n \geq 0} V_2((M_N^P - M_{2^n}^P) \mathcal{F}^{-1}(\Xi_n \hat{f}) : N \in [2^n, 2^{n+1}))^2 \right)^{1/2} \\ + \left(\sum_{n \geq 0} V_2(\mathcal{F}^{-1}((M_N^P - M_{2^n}^P)(1 - \Xi_n) \hat{f}) : N \in [2^n, 2^{n+1}))^2 \right)^{1/2}.$$

The last sum corresponds to the highly oscillatory behaviour of the multiplier m_N . Therefore, invoking inequality (2.8), Weyl's inequality in Theorem 3.1 and (1.9), we are able to prove that the last term in (1.13) is bounded on $\ell^p(\mathbb{Z}^d)$, see Section 5.

To bound the first term in (1.13) we introduce a tool reminiscent of the Littlewood-Paley theory. Now as opposed to the continous theory, for discrete operators there is no known analogue of the square functions of Littlewood-Paley that give us decisive control of the operators in question. However as a start in this direction we consider in Section 5, (see Theorem 5.1), the following family of projections:

$$(1.14) \quad \Delta_{n,s}^j(\xi) = \sum_{a/q \in \mathcal{U}_{(s+1)l} \setminus \mathcal{U}_{sl}} (\eta(\mathcal{E}_{n+j}(\xi - a/q)) - \eta(\mathcal{E}_{n+j+1}(\xi - a/q))) \eta(\mathcal{E}_s(\xi - a/q)).$$

and using Theorem 3.2 we will be able to show that for each $p \in (1, \infty)$ there is a constant $C > 0$ such that

$$(1.15) \quad \left\| \left(\sum_{n \in \mathbb{Z}} |\mathcal{F}^{-1}(\Delta_{n,s}^j \hat{f})|^2 \right)^{1/2} \right\|_{\ell^p} \leq C \log(s+2) \|f\|_{\ell^p}.$$

uniformly in $j \in \mathbb{Z}$. Estimate (1.15) can be thought as a discrete counterpart of Littlewood-Paley inequality and is essential in our further purposes. Thanks to (1.15) we reduce the problem to showing that

$$(1.16) \quad \sum_{s \geq 0} \sum_{j \in \mathbb{Z}} \left\| \left(\sum_{n \geq \max\{s, j, -j\}} V_2((M_N - M_{2^n}) \mathcal{F}^{-1}(\Delta_{n,s}^j \hat{f}) : N \in [2^n, 2^{n+1}))^2 \right)^{1/2} \right\|_{\ell^p} \leq C_p \|f\|_{\ell^p}.$$

In view of inequality (2.6) and (1.15) we show that the inner norm in (1.16) is dominated for every $p \in (1, \infty)$ by $C_p 2^{-\varepsilon_p |j|} (s+1)^{-2} \|f\|_{\ell^p}$ for some $\varepsilon_p > 0$. An important intermediate step in establishing this bound are the vector-valued estimates in [15]

$$\left\| \left(\sum_{t \in \mathbb{N}} \sup_{N \in \mathbb{N}} |M_N^{\mathcal{P}} f_t|^2 \right)^{1/2} \right\|_{\ell^p} \leq C_p \left\| \left(\sum_{t \in \mathbb{N}} |f_t|^2 \right)^{1/2} \right\|_{\ell^p}.$$

The idea of using vector-valued inequalities allows us to overcome many technical difficulties and, as far as we know, has not been used in this context before. We also employ this idea in the continuous setup and provide a new proof of short variations estimates for the operators of Radon type in [10]. We refer to the Appendix for more details.

The rest of the paper is organized as follows. In Section 2 we collected necessary numerical inequalities which give relation between r -variational seminorms and various square functions and even more general objects, see especially Lemma 2.1 and Lemma 2.2 and inequality (2.8). All these results are important building blocks in our approach. Finally, we propose some lifting lemma (see Lemma 2.3) which allows us to replace any polynomial mapping \mathcal{P} by the canonical polynomial mapping \mathcal{Q} which has all coefficients equal to 1. This guarantees that our further bounds will be independent of coefficients of the underlying polynomial mapping.

In Section 3 we recall further results whose proofs were given in [15]. Theorem 3.1 is a variant of multidimensional Weyl's sum estimates with logarithmic decay. We also include some basic tools which allow us to efficiently compare discrete $\|\cdot\|_{\ell^p}$ norms with continuous $\|\cdot\|_{L^p}$ norms. Finally, Theorem 3.2 is a major step towards proving (1.9) and (1.15). This theorem originates in Ionescu and Wainger paper [8] with $(\log N)^D$ loss where $D > 0$ is a large power. This is a deep result which uses the most sophisticated tools developed to date in this area.

In Section 4 we state Theorem 4.1 which is the main result of this Section. However, we omit the proof, since it can be deduced from the methods of proof of Theorem B from [15] by simply replacing the supremum norm by the long r -variational seminorm; or it can be completed following the scheme of the proof from Section 6, which contains long r -variational estimates for the operator $T_N^{\mathcal{P}}$. We have decided to provide a complete proof of long r -variational estimates for the operator $T_N^{\mathcal{P}}$, since there are some subtle differences which did not occur in [15] where the maximal function associated with $T_N^{\mathcal{P}}$ was studied, and this would cause some unnecessary confusions.

In Section 5 and Section 7 we provide detailed proofs of short variations estimates for operators $M_N^{\mathcal{P}}$ and $T_N^{\mathcal{P}}$ respectively.

Finally, in the Appendix, which is self-contained, we give a new proof of strong r -variational estimates for the operators of Radon type in the continuous setting, which are needed above. There are two novel aspects of our proof. The first concerns a different approach to Lépingle's inequality for martingales and is based on Theorem A.1 which is a new ingredient here. The second aspect concerns the estimates of short variations which now are based to a large extent on vector-valued estimates for operators of Radon type, which have been recently obtained in [15]. For the reader's convenience we provide details and describe this method in the context of dyadic martingales on homogeneous spaces; however, the methods are general enough to be applicable in a boarder context.

Finally, let us emphasize the following.

Remark 1.1. The methods of the proof of Theorem A and Theorem B allow us to extend inequalities (1.4) and (1.5) and establish the following.

Theorem D. For every $p \in (1, \infty)$ and $r \in (2, \infty)$ there is $C_{p,r} > 0$ such that for all $f \in \ell^p(\mathbb{Z}^{d_0})$

$$(1.17) \quad \|V_r(M_t^{\mathcal{P}} f : t > 0)\|_{\ell^p} + \|V_r(T_t^{\mathcal{P}} f : t > 0)\|_{\ell^p} \leq C_{p,r} \|f\|_{\ell^p}.$$

Moreover, the constant $C_{p,r} \leq C_p \frac{r}{r-2}$ for some $C_p > 0$ which is independent of the coefficients of the polynomial mapping \mathcal{P} .

The set of integers in the definition of r -variations has been replaced by the set $(0, \infty)$. The proof of Theorem D is presented in Section 8.

Remark 1.2. The methods of the proof of Theorem A and Theorem B give more general results. Namely, assume that G is an open bounded convex subset of \mathbb{R}^k containing the origin, and define for any $\lambda > 0$

$$G_\lambda = \{x \in \mathbb{R}^k : \lambda^{-1}x \in G\}$$

Define also $\mathbb{G}_\lambda = \{x \in \mathbb{Z}^k : \lambda^{-1}x \in G\}$ for any $\lambda > 0$. Then the inequality from Theorem A remains valid for the averaging operators (1.1) defined with \mathbb{G}_N rather than \mathbb{B}_N . Furthermore, if we assume that the

cancellation condition (1.2) holds with $G_{\lambda_2} \setminus G_{\lambda_1}$ instead of $B_{\lambda_2} \setminus B_{\lambda_1}$ then the conclusion of Theorem B remains valid for the singular truncated Radon transforms (1.3) defined with the summation taken over \mathbb{G}_N rather than \mathbb{B}_N .

1.1. Notation. Throughout the whole article, unless otherwise stated, we will write $A \lesssim B$ ($A \gtrsim B$) if there is an absolute constant $C > 0$ such that $A \leq CB$ ($A \geq CB$). Moreover, $C > 0$ will stand for a large positive constant whose value may vary from occurrence to occurrence. If $A \lesssim B$ and $A \gtrsim B$ hold simultaneously then we will write $A \simeq B$. We will denote $A \lesssim_\delta B$ ($A \gtrsim_\delta B$) to indicate that the constant $C > 0$ depends on some $\delta > 0$. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $N \in \mathbb{N}$ we set

$$\mathbb{N}_N = \{1, 2, \dots, N\}, \quad \text{and} \quad \mathbb{Z}_N = \{-N, \dots, -1, 0, 1, \dots, N\}.$$

For a vector $x \in \mathbb{R}^d$ we will use the following norms

$$|x|_\infty = \max\{|x_j| : 1 \leq j \leq d\}, \quad \text{and} \quad |x| = \left(\sum_{j=1}^d |x_j|^2 \right)^{1/2}.$$

If γ is a multi-index from \mathbb{N}_0^k then $|\gamma| = \gamma_1 + \dots + \gamma_k$. Although we use $|\cdot|$ for the length of a multi-index $\gamma \in \mathbb{N}_0^k$ and the Euclidean norm of $x \in \mathbb{R}^d$, their meaning will be always clear from the context and it will cause no confusions in the sequel. Finally, let $\mathcal{D} = \{2^n : n \in \mathbb{N}_0\}$ denote the set of dyadic numbers.

2. PRELIMINARIES

2.1. Variational norm. Let $1 \leq r < \infty$. For each sequence $(a_j : j \in A)$ of complex numbers, where $A \subseteq \mathbb{Z}$ we define r -variational seminorm by

$$V_r(a_j : j \in A) = \sup_{\substack{k_0 < k_1 < \dots < k_J \\ k_j \in A}} \left(\sum_{j=1}^J |a_{k_j} - a_{k_{j-1}}|^r \right)^{1/r}$$

where the supremum is taken over all finite increasing sequences of integers $k_0 < k_1 < \dots < k_J$. The function $r \mapsto V_r(a_j : j \in A)$ is non-increasing and satisfies

$$(2.1) \quad \sup_{j \in A} |a_j| \leq 2V_r(a_j : j \in A) + |a_{j_0}|$$

where j_0 is an arbitrary element of A . For any subset $B \subseteq A$ we have

$$V_r(a_j : j \in B) \leq V_r(a_j : j \in A).$$

Moreover, if $-\infty \leq u < w < v \leq \infty$ then

$$(2.2) \quad V_r(a_j : u \leq j < v) \leq 2 \sup_{u \leq j < v} |a_j| + V_r(a_j : u \leq j < w) + V_r(a_j : w \leq j < v).$$

For $r \geq 2$ we also have

$$(2.3) \quad V_r(a_j : j \in A) \leq 2 \left(\sum_{j \in A} |a_j|^2 \right)^{1/2}.$$

The next lemma will be critical in our further investigations. See also Lemma 2.2 in [15].

Lemma 2.1. *If $r \in [2, \infty)$ then for any sequence $(a_j : 0 \leq j \leq 2^s)$ of complex numbers we have*

$$(2.4) \quad V_r(a_j : 0 \leq j \leq 2^s) \leq \sqrt{2} \sum_{i=0}^s \left(\sum_{j=0}^{2^{s-i}-1} |a_{(j+1)2^i} - a_{j2^i}|^2 \right)^{1/2}.$$

Proof. Let us observe that any interval $[m, n)$ for $m, n \in \mathbb{N}$ such that $0 \leq m < n \leq 2^s$, is a finite disjoint union of dyadic subintervals, i.e. intervals belonging to some \mathcal{I}_i for $0 \leq i \leq s$, where

$$\mathcal{I}_i = \{[j2^i, (j+1)2^i) : 0 \leq j \leq 2^{s-i} - 1\}$$

and such that each length appears at most twice. For the proof, we set $m_0 = m$. Having chosen m_p we select m_{p+1} in such a way that $[m_p, m_{p+1})$ is the longest dyadic interval starting at m_p and contained inside $[m_p, n)$. If the lengths of the selected dyadic intervals increase then we are done. Otherwise, there is p such that $m_{p+1} - m_p \geq m_{p+2} - m_{p+1}$. We show that this implies $m_{p+2} - m_{p+1} > m_{p+3} - m_{p+2}$. Suppose for a contradiction that, $m_{p+2} - m_{p+1} \leq m_{p+3} - m_{p+2}$. Then

$$[m_{p+1}, 2m_{p+2} - m_{p+1}) \subseteq [m_{p+1}, m_{p+3}).$$

Therefore, it is enough to show that $2(m_{p+2} - m_{p+1})$ divides m_{p+1} . It is clear in the case $m_{p+1} - m_p > m_{p+2} - m_{p+1}$. If $m_{p+1} - m_p = m_{p+2} - m_{p+1}$ then, by the maximality of $[m_p, m_{p+1})$, $2(m_{p+2} - m_{p+1})$ cannot divide m_p , thus divides m_{p+1} .

Next, let $k_0 < k_1 < \dots < k_J \leq 2^s$ be any increasing sequence. For each $j \in \{0, \dots, J-1\}$ we may write

$$[k_j, k_{j+1}) = \bigcup_{p=0}^{P_j} [u_p^j, u_{p+1}^j)$$

for some $P_j \geq 1$ where each interval $[u_p^j, u_{p+1}^j)$ is dyadic. Then

$$|a_{k_{j+1}} - a_{k_j}| \leq \sum_{p=0}^{P_j} |a_{u_{p+1}^j} - a_{u_p^j}| = \sum_{i=0}^s \sum_{p: [u_p^j, u_{p+1}^j) \in \mathcal{I}_i} |a_{u_{p+1}^j} - a_{u_p^j}|.$$

Hence, by Minkowski's inequality

$$\begin{aligned} \left(\sum_{j=0}^{J-1} |a_{k_{j+1}} - a_{k_j}|^2 \right)^{1/2} &\leq \left(\sum_{j=0}^{J-1} \left(\sum_{i=0}^s \sum_{p: [u_p^j, u_{p+1}^j) \in \mathcal{I}_i} |a_{u_{p+1}^j} - a_{u_p^j}| \right)^2 \right)^{1/2} \\ &\leq \sum_{i=0}^s \left(\sum_{j=0}^{J-1} \left(\sum_{p: [u_p^j, u_{p+1}^j) \in \mathcal{I}_i} |a_{u_{p+1}^j} - a_{u_p^j}| \right)^2 \right)^{1/2}. \end{aligned}$$

Since for a given $i \in \{0, 1, \dots, 2^s\}$ and $j \in \{0, 1, \dots, J-1\}$ the inner sums contain at most two elements we obtain

$$\left(\sum_{j=0}^{J-1} |a_{k_{j+1}} - a_{k_j}|^2 \right)^{1/2} \leq \sqrt{2} \sum_{i=0}^s \left(\sum_{j=0}^{J-1} \sum_{p: [u_p^j, u_{p+1}^j) \in \mathcal{I}_i} |a_{u_{p+1}^j} - a_{u_p^j}|^2 \right)^{1/2}$$

which is bounded by the right-hand side of (2.4). \square

A long variation seminorm V_r^L of a sequence $(a_j : j \in A)$, is given by

$$V_r^L(a_j : j \in A) = V_r(a_j : j \in A \cap \mathcal{D}).$$

A short variation seminorm V_r^S is given by

$$V_r^S(a_j : j \in A) = \left(\sum_{n \geq 0} V_r(a_j : j \in A_n) \right)^{1/r}$$

where $A_n = A \cap [2^n, 2^{n+1})$. Then

$$(2.5) \quad V_r(a_j : j \in \mathbb{N}) \lesssim V_r^L(a_j : j \in \mathbb{N}) + V_r^S(a_j : j \in \mathbb{N}).$$

The next lemma will be used in the estimates for short variations. It illustrates the ideas which have been explored several times (see [9], or recently [11, 22]).

Lemma 2.2. *Let $u, v \in \mathbb{N}$, $u < v$. For any integer $h \in \{1, \dots, v-u\}$ there is a strictly increasing sequence of integers $(t_j : 0 \leq j \leq h)$ with $t_0 = u$ and $t_h = v$ such that for every $r \in [1, \infty)$*

$$(2.6) \quad V_r(a_j : u \leq j \leq v) \lesssim \left(\sum_{j=0}^h |a_{t_j}|^r \right)^{1/r} + \left(\sum_{j=0}^{h-1} \left(\sum_{k=t_j}^{t_{j+1}-1} |a_{k+1} - a_k| \right)^r \right)^{1/r}.$$

Moreover, if $p \geq r$ then

$$\begin{aligned} (2.7) \quad &\left(\sum_{j=0}^h |a_{t_j}|^r \right)^{1/r} + \left(\sum_{j=0}^{h-1} \left(\sum_{k=t_j}^{t_{j+1}-1} |a_{k+1} - a_k| \right)^r \right)^{1/r} \\ &\lesssim h^{1/r-1/p} \left(\sum_{j=0}^h |a_{t_j}|^p \right)^{1/p} + h^{1/r-1}(v-u)^{1-1/p} \left(\sum_{j=u}^{v-1} |a_{j+1} - a_j|^p \right)^{1/p}. \end{aligned}$$

The implicit constants in (2.6) and (2.7) are independent of h, u and v .

Proof. Fix $h \in \{1, \dots, v-u\}$ and choose a sequence $(t_j : 1 \leq j \leq h)$ such that $t_0 = u$, $t_h = v$ and $|t_{j+1} - t_j| \simeq (v-u)/h$. Then

$$\begin{aligned} V_r(a_j : u \leq j \leq v) &\lesssim \left(\sum_{j=0}^h |a_{t_j}|^r \right)^{1/r} + \left(\sum_{j=0}^{h-1} V_r(a_k : t_j \leq k \leq t_{j+1})^r \right)^{1/r} \\ &\lesssim \left(\sum_{j=0}^h |a_{t_j}|^r \right)^{1/r} + \left(\sum_{j=0}^{h-1} \left(\sum_{k=t_j}^{t_{j+1}-1} |a_{k+1} - a_k| \right)^r \right)^{1/r}. \end{aligned}$$

If $p \geq r$ then by Hölder's inequality the last sum can be dominated by

$$\begin{aligned} &\left(\sum_{j=0}^h |a_{t_j}|^r \right)^{1/r} + \left(\sum_{j=0}^{h-1} \left(\sum_{k=t_j}^{t_{j+1}-1} |a_{k+1} - a_k| \right)^r \right)^{1/r} \\ &\lesssim h^{1/r-1/p} \left(\sum_{j=0}^h |a_{t_j}|^p \right)^{1/p} + h^{1/r-1/p} \left(\sum_{j=0}^{h-1} \left(\sum_{k=t_j}^{t_{j+1}-1} |a_{k+1} - a_k| \right)^p \right)^{1/p} \\ &\lesssim h^{1/r-1/p} \left(\sum_{j=0}^h |a_{t_j}|^p \right)^{1/p} + h^{1/r-1/p} \left(\sum_{j=0}^{h-1} (t_{j+1} - t_j)^{p(1-1/p)} \left(\sum_{k=t_j}^{t_{j+1}-1} |a_{k+1} - a_k|^p \right) \right)^{1/p} \\ &\lesssim h^{1/r-1/p} \left(\sum_{j=0}^h |a_{t_j}|^p \right)^{1/p} + h^{1/r-1/p} (v-u)^{1-1/p} \left(\sum_{j=u}^{v-1} |a_{j+1} - a_j|^p \right)^{1/p} \end{aligned}$$

and this completes the proof of the lemma. \square

We observe that, if $1 \leq r \leq p$ and $(f_j : j \in \mathbb{N})$ is a sequence of functions in $\ell^p(\mathbb{Z}^d)$ and $v-u \geq 2$ then

$$(2.8) \quad \|V_r(f_j : j \in [u, v])\|_{\ell^p} \lesssim \max \{ \mathbf{U}_p, (v-u)^{1/r} \mathbf{U}_p^{1-1/r} \mathbf{V}_p^{1/r} \}$$

where

$$\mathbf{U}_p = \max_{u \leq j \leq v} \|f_j\|_{\ell^p}, \quad \text{and} \quad \mathbf{V}_p = \max_{u \leq j < v} \|f_{j+1} - f_j\|_{\ell^p}.$$

Indeed, let

$$h = \lceil (v-u) \mathbf{V}_p / (4 \mathbf{U}_p) \rceil.$$

Then $h \in [1, v-u]$. If $h \geq 2$, by Lemma 2.2, we have

$$\|V_r(f_j : u \leq j \leq v)\|_{\ell^p} \lesssim h^{1/r} \mathbf{U}_p + h^{1/r-1} (v-u) \mathbf{V}_p \lesssim (v-u)^{1/r} \mathbf{U}_p^{1-1/r} \mathbf{V}_p^{1/r}.$$

If $h = 1$ then $\mathbf{V}_p \lesssim (v-u)^{-1} \mathbf{U}_p$ and hence

$$\|V_r(f_j : u \leq j \leq v)\|_{\ell^p} \lesssim \mathbf{U}_p.$$

2.2. Lifting lemma. Let $\mathcal{P} = (\mathcal{P}_1, \dots, \mathcal{P}_{d_0}) : \mathbb{Z}^k \rightarrow \mathbb{Z}^{d_0}$ be a mapping whose components \mathcal{P}_j are integer-valued polynomials on \mathbb{Z}^k such that $\mathcal{P}_j(0) = 0$. We set

$$N_0 = \max \{ \deg \mathcal{P}_j : 1 \leq j \leq d_0 \}.$$

It is convenient to work with the set

$$\Gamma = \{ \gamma \in \mathbb{Z}^k \setminus \{0\} : 0 \leq \gamma_j \leq N_0 \text{ for each } j = 1, \dots, k \}$$

with the lexicographic order. Then each \mathcal{P}_j can be expressed as

$$\mathcal{P}_j(x) = \sum_{\gamma \in \Gamma} c_j^\gamma x^\gamma$$

for some $c_j^\gamma \in \mathbb{Z}$. Let us denote by d the cardinality of the set Γ . We identify \mathbb{R}^d with the space of all vectors whose coordinates are labelled by multi-indices $\gamma \in \Gamma$. Let A be a diagonal $d \times d$ matrix such that

$$(Av)_\gamma = |\gamma| v_\gamma.$$

For $t > 0$ we set

$$t^A = \exp(A \log t)$$

i.e. $t^A x = (t^{|\gamma|} x_\gamma : \gamma \in \Gamma)$ for any $x \in \mathbb{R}^d$. Next, we introduce the *canonical* polynomial mapping

$$\mathcal{Q} = (\mathcal{Q}_\gamma : \gamma \in \Gamma) : \mathbb{Z}^k \rightarrow \mathbb{Z}^d$$

where $\mathcal{Q}_\gamma(x) = x^\gamma$ and $x^\gamma = x_1^{\gamma_1} \cdots x_k^{\gamma_k}$. The coefficients $(c_j^\gamma : \gamma \in \Gamma, j \in \{1, \dots, d_0\})$ define a linear transformation $L : \mathbb{R}^d \rightarrow \mathbb{R}^{d_0}$ such that $L\mathcal{Q} = \mathcal{P}$. Indeed, it is enough to set

$$(Lv)_j = \sum_{\gamma \in \Gamma} c_j^\gamma v_\gamma$$

for each $j \in \{1, \dots, d_0\}$ and $v \in \mathbb{R}^d$. The next lemma, inspired by the continuous analogue (see [5] or [19, p. 515]) reduces proofs of Theorem A and Theorem B to the canonical polynomial mapping. See also Lemma 2.1 in [15].

Lemma 2.3. *Let $R_N^{\mathcal{P}}$ be one of the operators $M_N^{\mathcal{P}}$ or $T_N^{\mathcal{P}}$. Suppose that for some $p \in (1, \infty)$ and $r \in (2, \infty)$*

$$\|\mathcal{V}_r(R_N^{\mathcal{Q}}f : N \in \mathbb{N})\|_{\ell^p(\mathbb{Z}^d)} \leq C_{p,r} \|f\|_{\ell^p(\mathbb{Z}^d)}.$$

Then

$$(2.9) \quad \|\mathcal{V}_r(R_N^{\mathcal{P}}f : N \in \mathbb{N})\|_{\ell^p(\mathbb{Z}^{d_0})} \leq C_{p,r} \|f\|_{\ell^p(\mathbb{Z}^d)}.$$

Proof. For the proof we refer to [15]. □

From now on M_N and T_N will denote the operators defined for the canonical polynomial mapping \mathcal{Q} , i.e. $M_N = M_N^{\mathcal{Q}}$ and $T_N = T_N^{\mathcal{Q}}$.

3. FURTHER TOOLS

3.1. Gaussian sums. For $q \in \mathbb{N}$ let us define

$$A_q = \{a \in \mathbb{N}_q^d : \gcd(q, (a_\gamma : \gamma \in \Gamma))\}.$$

Next, for $q \in \mathbb{N}$ and $a \in A_q$ we define the *Gaussian sum*

$$G(a/q) = q^{-k} \sum_{y \in \mathbb{N}_q^k} e^{2\pi i(a/q) \cdot \mathcal{Q}(y)}.$$

Let us observe that, by the multi-dimensional variant of Weyl's inequality (see [20, Proposition 3]), there exists $\delta > 0$ such that

$$(3.1) \quad |G(a/q)| \lesssim q^{-\delta}.$$

3.2. Weyl's estimates. Let P be a polynomial in \mathbb{R}^k of degree $d \in \mathbb{N}$ such that

$$P(x) = \sum_{0 < |\gamma| \leq d} \xi_\gamma x^\gamma.$$

Given $N \geq 1$, let Ω_N be a convex set such that

$$\Omega_N \subseteq B_{cN}(x_0)$$

for some $x_0 \in \mathbb{R}^k$ and $c > 0$, where $B_r(x_0) = \{x \in \mathbb{R}^k : |x - x_0| \leq r\}$. We define

$$S_N = \sum_{n \in \Omega_N \cap \mathbb{Z}^k} e^{2\pi i P(n)} \varphi(n)$$

where $\varphi : \mathbb{R}^k \rightarrow \mathbb{C}$ is a $\mathcal{C}^1(\mathbb{R}^k)$ function which for some $C > 0$ satisfies

$$|\varphi(x)| \leq C, \quad \text{and} \quad |\nabla \varphi(x)| \leq C(1 + |x|)^{-1}.$$

In [15] we proved the following refinement of multi-dimensional Weyl's inequality.

Theorem 3.1. *Assume that there is a multi-index γ_0 such that $0 < |\gamma_0| \leq d$ and*

$$\left| \xi_{\gamma_0} - \frac{a}{q} \right| \leq \frac{1}{q^2}$$

for some integers a, q such that $0 \leq a \leq q$ and $(a, q) = 1$. Then for any $\alpha > 0$ there is $\beta_\alpha > 0$ so that, for any $\beta \geq \beta_\alpha$, if

$$(\log N)^\beta \leq q \leq N^{|\gamma_0|} (\log N)^{-\beta}$$

then there is a constant $C > 0$

$$|S_N| \leq CN^k (\log N)^{-\alpha}.$$

The implied constant C is independent of N .

3.3. Transference principle. Let \mathcal{F} denote the Fourier transform on \mathbb{R}^d defined for any function $f \in L^1(\mathbb{R}^d)$ as

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}^d} f(x) e^{2\pi i \xi \cdot x} dx.$$

If $f \in \ell^1(\mathbb{Z}^d)$ we set

$$\hat{f}(\xi) = \sum_{x \in \mathbb{Z}^d} f(x) e^{2\pi i \xi \cdot x}.$$

To simplify the notation we denote by \mathcal{F}^{-1} the inverse Fourier transform on \mathbb{R}^d or the inverse Fourier transform on \mathbb{T}^d (Fourier coefficients), depending on the context.

Let $\eta : \mathbb{R}^d \rightarrow \mathbb{R}$ be a smooth function such that $0 \leq \eta(x) \leq 1$ and

$$\eta(x) = \begin{cases} 1 & \text{for } |x| \leq 1/(16d), \\ 0 & \text{for } |x| \geq 1/(8d). \end{cases}$$

Remark 3.1. We may additionally assume that η is a convolution of two non-negative smooth functions ϕ and ψ with compact supports contained inside $(-1/(8d), 1/(8d))^d$.

Let $(\Theta_N : N \in \mathbb{N})$ be a sequence of multipliers on \mathbb{R}^d with a property that for each $p \in (1, \infty)$ and $r \in (2, \infty)$ there is a constant $\mathbf{B}_{p,r} > 0$ such that for any $f \in L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$

$$(3.2) \quad \|V_r(\mathcal{F}^{-1}(\Theta_N \mathcal{F}f) : N \in \mathbb{N})\|_{L^p} \leq \mathbf{B}_{p,r} \|f\|_{L^p}.$$

Moreover, $\mathbf{B}_{p,r} \leq \mathbf{B}_p \frac{r}{r-2}$ for some $\mathbf{B}_p > 0$. In fact we will be only interested in the multipliers which are discussed in the Appendix, see Theorem A.1 and Theorem A.2.

We assume that \mathcal{R} is a diagonal $d \times d$ matrix with positive entries $(r_\gamma : \gamma \in \Gamma)$ such that $\inf_{\gamma \in \Gamma} r_\gamma \geq h$ for some $h > 0$. In [15] we proved, in particular, the following version of the transference principle.

Proposition 3.1. *Under assumption (3.2) for each $p \in (1, \infty)$ and $r \in (2, \infty)$ there is a constant $C > 0$ such that for each $Q \in \mathbb{N}$ and $h \geq 2Q^{d+1}$ and any $m \in \mathbb{N}_Q^k$ we have*

$$\|\mathcal{V}_r(\mathcal{F}^{-1}(\Theta_N \eta(\mathcal{R} \cdot) \hat{f})(Qx + m) : N \in \mathbb{N})\|_{\ell^p(x)} \leq C \mathbf{B}_{p,r} \|\mathcal{F}^{-1}(\eta(\mathcal{R} \cdot) \hat{f})(Qx + m)\|_{\ell^p(x)}.$$

See also the discussion of sampling in [14, Proposition 2.1 and Corollary 2.5].

3.4. Ionescu–Wainger type multipliers. We now introduce necessary notation to define Ionescu and Wainger type multipliers. To fix notation set $\rho > 0$ and for every $N \in \mathbb{N}$, let us define $N_0 = \lfloor N^{\rho/2} \rfloor + 1$, moreover let $Q_0 = (N_0!)^D$ where $D = D_\rho = \lfloor 2/\rho \rfloor + 1$. Let \mathbb{P} denote the set of all prime numbers and $\mathbb{P}_N = \mathbb{P} \cap (N_0, N]$. For any $k \in \mathbb{N}_D$ and $V \subseteq \mathbb{P}_N$ we define

$$\Pi_k(V) = \{p_1^{\gamma_1} \cdots p_k^{\gamma_k} : \gamma_l \in \mathbb{N}_D \text{ and } p_l \in V \text{ are distinct for all } 1 \leq l \leq k\}.$$

Therefore, $\Pi_{k_1}(V) \cap \Pi_{k_2}(V) = \emptyset$ if $k_1 \neq k_2$ and

$$\Pi(V) = \bigcup_{k \in \mathbb{N}_D} \Pi_k(V)$$

is the set of all products of primes factors from V of length at most D , at powers between 1 and D .

It is easy to see, now, that every integer $q \in \mathbb{N}_N$ can be uniquely written as $q = Q \cdot w$ where $Q|Q_0$ and $w \in \Pi(\mathbb{P}_N) \cup \{1\}$. Moreover, for sufficiently large $N \in \mathbb{N}$

$$q = Q \cdot w \leq Q_0 \cdot w \leq (N_0!)^D N^{D^2} \leq e^{N^\rho}$$

thus for the set

$$P_N = \{q = Q \cdot w : Q|Q_0 \text{ and } w \in \Pi(\mathbb{P}_N) \cup \{1\}\}$$

we have $\mathbb{N}_N \subseteq P_N \subseteq \mathbb{N}_{e^{N^\rho}}$. Moreover, if $N_1 \leq N_2$ then $P_{N_1} \subseteq P_{N_2}$. For any $S \subseteq \mathbb{N}$ define

$$\mathcal{R}(S) = \{a/q \in \mathbb{Q}^d \cap \mathbb{T}^d : a \in A_q \text{ and } q \in S\}.$$

We will assume that for every $p \in (1, \infty)$ there is a constant $\mathbf{A}_p > 0$

$$(3.3) \quad \|\mathcal{F}^{-1}(\Theta \mathcal{F}f)\|_{L^p} \leq \mathbf{A}_p \|f\|_{L^p}.$$

For each $N \in \mathbb{N}$ we define new periodic multipliers

$$\Delta_N(\xi) = \sum_{a/q \in \mathcal{U}_N} \Theta(\xi - a/q) \eta_N(\xi - a/q)$$

where $\eta_N(\xi) = \eta(\mathcal{E}_N^{-1}\xi)$ and \mathcal{E}_N is a diagonal $d \times d$ matrix with positive entries $(\varepsilon_\gamma : \gamma \in \Gamma)$ such that $\varepsilon_\gamma \leq e^{-N^{2\rho}}$ and

$$(3.4) \quad \mathcal{U}_N = \mathcal{R}(P_N)$$

Furthermore, if $N_1 \leq N_2$ then $\mathcal{U}_{N_1} \subseteq \mathcal{U}_{N_2}$. The main result of this subsection is the following.

Theorem 3.2. *Let Θ be a multiplier on \mathbb{R}^d obeying (3.3). Then for every $\rho > 0$ and $p \in (1, \infty)$ there is a constant $C_{\rho,p} > 0$ such that for any $N \in \mathbb{N}$ and $f \in \ell^p(\mathbb{Z}^d)$*

$$\|\mathcal{F}^{-1}(\Delta_N \hat{f})\|_{\ell^p} \leq C_{\rho,p} \mathbf{A}_p(\log N) \|f\|_{\ell^p}.$$

Theorem 3.2 inspired by the ideas of Ionescu and Wainger from [8] was proven in [15].

4. LONG VARIATION ESTIMATES FOR AVERAGING OPERATORS

For any function $f : \mathbb{Z}^d \rightarrow \mathbb{C}$ with a finite support we have

$$M_N f(x) = K_N * f(x)$$

where K_N is a kernel defined by

$$K_N(x) = |\mathbb{B}_N|^{-1} \sum_{y \in \mathbb{B}_N} \delta_{\mathcal{Q}(y)}$$

and δ_y denotes Dirac's delta at $y \in \mathbb{Z}^k$. Let m_N denote the discrete Fourier transform of K_N , i.e.

$$m_N(\xi) = |\mathbb{B}_N|^{-1} \sum_{y \in \mathbb{B}_N} e^{2\pi i \xi \cdot \mathcal{Q}(y)}.$$

Finally, we define

$$\Phi_N(\xi) = |B_1|^{-1} \int_{B_1} e^{2\pi i \xi \cdot \mathcal{Q}(Ny)} dy.$$

Using a multi-dimensional version of van der Corput lemma (see [19, 2]) we may estimate

$$(4.1) \quad |\Phi_N(\xi)| \lesssim \min \{1, |N^A \xi|_\infty^{-1/d}\}.$$

Additionally, we have

$$(4.2) \quad |\Phi_N(\xi) - 1| \lesssim \min \{1, |N^A \xi|_\infty\}.$$

Proposition 4.1. *There is a constant $C > 0$ such that for every $N \in \mathbb{N}$ and for every $\xi \in [-1/2, 1/2]^d$ satisfying*

$$\left| \xi_\gamma - \frac{a_\gamma}{q} \right| \leq L_1^{-|\gamma|} L_2$$

for all $\gamma \in \Gamma$, where $1 \leq q \leq L_3 \leq N^{1/2}$, $a \in A_q$, $L_1 \geq N$ and $L_2 \geq 1$ we have

$$(4.3) \quad |m_N(\xi) - G(a/q) \Phi_N(\xi - a/q)| \leq C \left(L_3 N^{-1} + L_2 L_3 N^{-1} \sum_{\gamma \in \Gamma} (N/L_1)^{|\gamma|} \right) \lesssim L_2 L_3 N^{-1}.$$

Proof. Let $\theta = \xi - a/q$. For any $r \in \mathbb{N}_q^k$, if $y \equiv r \pmod{q}$ then for each $\gamma \in \Gamma$

$$\xi_\gamma y^\gamma \equiv \theta_\gamma y^\gamma + (a_\gamma/q) r^\gamma \pmod{1},$$

thus

$$\xi \cdot \mathcal{Q}(y) \equiv \theta \cdot \mathcal{Q}(y) + (a/q) \cdot \mathcal{Q}(r) \pmod{1}.$$

Therefore,

$$|\mathbb{B}_N|^{-1} \sum_{y \in \mathbb{B}_N} e^{2\pi i \xi \cdot \mathcal{Q}(y)} = q^{-k} \sum_{r \in \mathbb{N}_q^k} e^{2\pi i (a/q) \cdot \mathcal{Q}(r)} \cdot \left(q^k |\mathbb{B}_N|^{-1} \sum_{\substack{y \in \mathbb{Z}^k \\ |qy+r| \leq N}} e^{2\pi i \theta \cdot \mathcal{Q}(qy+r)} \right).$$

If $|qy+r|, |qy| \leq N$ then

$$|\theta \cdot \mathcal{Q}(qy+r) - \theta \cdot \mathcal{Q}(qy)| \lesssim |r| \sum_{\gamma \in \Gamma} |\theta_\gamma| \cdot N^{(|\gamma|-1)} \lesssim q \sum_{\gamma \in \Gamma} L_1^{-|\gamma|} L_2 N^{(|\gamma|-1)} \lesssim L_2 L_3 / N \sum_{\gamma \in \Gamma} (N/L_1)^{|\gamma|}.$$

Thus

$$|\mathbb{B}_N|^{-1} \sum_{y \in \mathbb{B}_N} e^{2\pi i \xi \cdot \mathcal{Q}(y)} = G(a/q) \cdot q^k |\mathbb{B}_N|^{-1} \sum_{\substack{y \in \mathbb{Z}^k \\ |qy| \leq N}} e^{2\pi i \theta \cdot \mathcal{Q}(qy)} + \mathcal{O}\left(q/N + L_2 L_3 / N \sum_{\gamma \in \Gamma} (N/L_1)^{|\gamma|}\right).$$

We have used the formula for the number of lattice points in the Euclidean ball, i.e. $|\mathbb{B}_N| = |B_1|N^k + \mathcal{O}(N^{k-1})$ as $N \rightarrow \infty$. Now we are going to replace the exponential sum on the right-hand side of the last display by the integral. By the mean value theorem, we obtain

$$\begin{aligned}
& \left| \sum_{\substack{y \in \mathbb{Z}^k \\ |qy| \leq N}} e^{2\pi i \theta \cdot \mathcal{Q}(qy)} - \int_{|qt| \leq N} e^{2\pi i \theta \cdot \mathcal{Q}(qt)} dt \right| \\
&= \left| \sum_{y \in \mathbb{Z}^k} e^{2\pi i \theta \cdot \mathcal{Q}(qy)} \mathbb{1}_{B_N}(qy) - \sum_{y \in \mathbb{Z}^k} \int_{y+(0,1]^k} e^{2\pi i \theta \cdot \mathcal{Q}(qt)} \mathbb{1}_{B_N}(qt) dt \right| \\
&= \left| \sum_{y \in \mathbb{Z}^k} \int_{(0,1]^k} e^{2\pi i \theta \cdot \mathcal{Q}(qy)} \mathbb{1}_{B_N}(qy) - e^{2\pi i \theta \cdot \mathcal{Q}(q(t+y))} \mathbb{1}_{B_N}(q(t+y)) dt \right| \\
&= \mathcal{O}\left((N/q)^{k-1} + (N/q)^k L_2 L_3 / N \sum_{\gamma \in \Gamma} (N/L_1)^{|\gamma|}\right).
\end{aligned}$$

This completes the proof of Proposition 4.1. \square

In Remark 1.2 we mentioned that Theorem A holds with the operators M_N defined with the sets \mathbb{G}_N instead of \mathbb{B}_N . Then we obtain analogous definitions of K_N , m_N and Φ_N with the sets \mathbb{G}_N and $G = G_1$ in place of the sets \mathbb{B}_N and B_1 respectively. All of the arguments remain unchanged apart from the proof of Proposition 4.1. Here we must proceed more delicately. However, [15, Proposition 3.1] used with the sets \mathbb{G}_N in place of the asymptotic formula for the number of lattice points in \mathbb{B}_N does the job and we obtain conclusion of the same type. Proposition 3.1 from [15] states that for a given convex set $\Omega \subseteq \mathbb{R}^k$ such that $B_{cr}(x'_0) \subseteq \Omega \subseteq B_r(x_0)$ for some $x_0, x'_0 \in \mathbb{R}^k$ and $c > 0$, we have that for any $1 \leq s \leq r$ the number of lattice points N_Ω in Ω of distance $< s$ from the boundary of Ω is $\mathcal{O}(sr^{k-1})$.

The main result of this section is the following.

Theorem 4.1. *For every $1 < p < \infty$ and $r \in (2, \infty)$ there is $C_p > 0$ such that for all $f \in \ell^p(\mathbb{Z}^{d_0})$*

$$(4.4) \quad \|V_r(M_{2^n} f : n \in \mathbb{N})\|_{\ell^p} \leq C_p \frac{r}{r-2} \|f\|_{\ell^p}.$$

As we have indicated, a complete proof of this theorem will not be given here. It will suffice to say that it uses Proposition 4.1 and follows the ideas [15, Section 6]. If one replaces the supremum norm occuring there by the variational norm V_r , we can then obtain (4.4). One can also follow closely the corresponding argument for the singular Radon transforms in Section 6.

5. SHORT VARIATION ESTIMATES FOR AVERAGING OPERATORS

According to (2.5) and the estimates for long variations from the previous section it remains to prove that for all $p \in (1, \infty)$ there is $C_p > 0$ such that for all $f \in \ell^p(\mathbb{Z}^d)$ with finite support we have

$$\left\| \left(\sum_{n \geq 0} V_2(M_N f : N \in [2^n, 2^{n+1})) \right)^{1/2} \right\|_{\ell^p} \leq C_p \|f\|_{\ell^p}.$$

For this purpose, fix the numbers $\chi > 0$ and $l \in \mathbb{N}$ whose precise values will be specified later, and let us introduce for every $n \in \mathbb{N}_0$ the multipliers

$$\Xi_n(\xi) = \sum_{a/q \in \mathcal{U}_{n,l}} \eta(2^{n(A-\chi I)}(\xi - a/q))$$

with $\mathcal{U}_{n,l}$ defined as in (3.4). Theorem 3.2 guarantees that for every $p \in (1, \infty)$

$$(5.1) \quad \|\mathcal{F}^{-1}(\Xi_n \hat{f})\|_{\ell^p} \lesssim \log(n+2) \|f\|_{\ell^p}.$$

The implicit constant in (5.1) depends on the parameter $\rho > 0$, which was fixed, see Section 3. However, from now on we will assume that $\rho > 0$ and the integer $l \geq 10$ are related by the equation

$$10\rho l = 1.$$

Observe that

$$(5.2) \quad \left\| \left(\sum_{n \geq 0} V_2(M_N f : N \in [2^n, 2^{n+1}))^2 \right)^{1/2} \right\|_{\ell^p} \\ \leq \left\| \left(\sum_{n \geq 0} V_2((M_N - M_{2^n})\mathcal{F}^{-1}(\Xi_n \hat{f}) : N \in [2^n, 2^{n+1}))^2 \right)^{1/2} \right\|_{\ell^p} \\ + \left\| \left(\sum_{n \geq 0} V_2((M_N - M_{2^n})\mathcal{F}^{-1}((1 - \Xi_n)\hat{f}) : N \in [2^n, 2^{n+1}))^2 \right)^{1/2} \right\|_{\ell^p}.$$

5.1. The estimate of the second norm in (5.2). We may assume without loss of generality, that $1 < r \leq \min\{2, p\}$, since r -variations are decreasing, and it suffices to show that

$$(5.3) \quad \|V_r((M_N - M_{2^n})\mathcal{F}^{-1}((1 - \Xi_n)\hat{f}) : N \in [2^n, 2^{n+1}))\|_{\ell^p} \lesssim (n+1)^{-2} \|f\|_{\ell^p}.$$

Appealing to (2.8) we immediately see that

$$(5.4) \quad \|V_r((M_N - M_{2^n})\mathcal{F}^{-1}((1 - \Xi_n)\hat{f}) : N \in [2^n, 2^{n+1}))\|_{\ell^p} \lesssim \max\{\mathbf{U}_p, 2^{n/r} \mathbf{U}_p^{1-1/r} \mathbf{V}_p^{1/r}\}$$

where

$$\mathbf{U}_p = \sup_{2^n \leq N \leq 2^{n+1}} \|(M_N - M_{2^n})\mathcal{F}^{-1}((1 - \Xi_n)\hat{f})\|_{\ell^p}$$

and

$$\mathbf{V}_p = \sup_{2^n \leq N < 2^{n+1}} \|(M_{N+1} - M_N)\mathcal{F}^{-1}((1 - \Xi_n)\hat{f})\|_{\ell^p}.$$

In view of (5.1) we see that

$$(5.5) \quad \mathbf{U}_p \lesssim \log(n+2) \|f\|_{\ell^p} \quad \text{and} \quad \mathbf{V}_p \lesssim 2^{-n} \log(n+2) \|f\|_{\ell^p}.$$

In fact we show that there is possible a refinement of these estimates for $p = 2$, which in turn improve the estimates from (5.5) for all $p \in (1, \infty)$ and finally one could conclude (5.3). Indeed, we claim that for big enough $\alpha > 0$, which will be specified later, and for all $n \in \mathbb{N}_0$ and $N \simeq 2^n$ we have

$$(5.6) \quad |(m_N(\xi) - m_{2^n}(\xi))(1 - \Xi_n(\xi))| \lesssim (n+1)^{-\alpha}.$$

This estimate will be a consequence of Theorem 3.1. To do so, by Dirichlet's principle we have for every $\gamma \in \Gamma$

$$\left| \xi_\gamma - \frac{a_\gamma}{q_\gamma} \right| \leq \frac{n^\beta}{q_\gamma 2^{n|\gamma|}}$$

where $1 \leq q_\gamma \leq n^{-\beta} 2^{n|\gamma|}$. In order to apply Theorem 3.1 we must show that there exists some $\gamma \in \Gamma$ such that $n^\beta \leq q_\gamma \leq n^{-\beta} 2^{n|\gamma|}$. Suppose for a contradiction that for every $\gamma \in \Gamma$ we have $1 \leq q_\gamma < n^\beta$. Then for some $q \leq \text{lcm}(q_\gamma : \gamma \in \Gamma) \leq n^{\beta d}$ we have

$$\left| \xi_\gamma - \frac{a'_\gamma}{q} \right| \leq \frac{n^\beta}{2^{n|\gamma|}}$$

where $\gcd(q, \gcd(a'_\gamma : \gamma \in \Gamma)) = 1$. Hence, taking $a' = (a'_\gamma : \gamma \in \Gamma)$ we have $a'/q \in \mathcal{U}_{n^l}$ provided that $\beta d < l$. On the other hand, if $1 - \Xi_n(\xi) \neq 0$ then for every $a'/q \in \mathcal{U}_{n^l}$ there exists $\gamma \in \Gamma$ such that

$$\left| \xi_\gamma - \frac{a'_\gamma}{q} \right| > \frac{1}{16d \cdot 2^{n(|\gamma| - \chi)}}.$$

Therefore, one obtains

$$2^{n\chi} < 16dn^\beta$$

which is not possible if n is sufficiently large. We have already shown that there exists some $\gamma \in \Gamma$ such that $n^\beta \leq q_\gamma \leq n^{-\beta} 2^{n|\gamma|}$ and consequently Theorem 3.1 yields

$$|m_N(\xi)| \lesssim (n+1)^{-\alpha}$$

provided that $1 - \Xi_n(\xi) \neq 0$ proving (5.6). We obtain

$$(5.7) \quad \mathbf{U}_2 \lesssim (1+n)^{-\alpha} \log(n+2) \|f\|_{\ell^2}.$$

Interpolating (5.7) with (5.5) we obtain

$$(5.8) \quad \mathbf{U}_p \lesssim (1+n)^{-c_p \alpha} \log(n+2) \|f\|_{\ell^p}.$$

for some $c_p > 0$. Now, by choosing $\alpha > 0$ and $l \in \mathbb{N}$ appropriately large we see that (5.8) combined with (5.4) easily imply (5.3).

5.2. The estimate of the first norm in (5.2). Note that for any $\xi \in \mathbb{T}^d$ such that

$$\left| \xi_\gamma - \frac{a_\gamma}{q} \right| \leq \frac{1}{8d \cdot 2^{n(|\gamma|-\chi)}}$$

for any $\gamma \in \Gamma$ with $1 \leq q \leq e^{n^{1/10}}$ we have

$$(5.9) \quad m_N(\xi) = G(a/q)\Phi_N(\xi - a/q) + q^{-\delta}E_{2^n}(\xi)$$

for every $N \in [2^n, 2^{n+1})$, where

$$(5.10) \quad |E_{2^n}(\xi)| \lesssim 2^{-n/2}.$$

These two properties (5.9) and (5.10) follow from Proposition 4.1 with $L_1 = 2^n$, $L_2 = (8d)^{-1}2^{n\chi}$ and $L_3 = e^{n^{1/10}}$, since

$$|E_{2^n}(\xi)| \lesssim q^\delta L_2 L_3 2^{-n} \lesssim e^{-n((1-\chi)\log 2 - 2n^{-9/10})} \lesssim 2^{-n/2}$$

which holds for sufficiently large $n \in \mathbb{N}$ when $\chi > 0$ is sufficiently small. Let us introduce for every $j, n \in \mathbb{N}_0$ the new multipliers

$$\Xi_n^j(\xi) = \sum_{a/q \in \mathcal{U}_{n^l}} \eta(2^{nA+jI}(\xi - a/q))$$

and note that

$$\begin{aligned} & \left\| \left(\sum_{n \geq 0} V_2((M_N - M_{2^n})\mathcal{F}^{-1}(\Xi_n \hat{f}) : N \in [2^n, 2^{n+1}))^2 \right)^{1/2} \right\|_{\ell^p} \\ & \leq \left\| \left(\sum_{n \geq 0} V_2((M_N - M_{2^n})\mathcal{F}^{-1} \left(\sum_{-\chi n \leq j < n} (\Xi_n^j - \Xi_n^{j+1}) \hat{f} \right) : N \in [2^n, 2^{n+1}))^2 \right)^{1/2} \right\|_{\ell^p} \\ & \quad + \left\| \left(\sum_{n \geq 0} V_2((M_N - M_{2^n})\mathcal{F}^{-1}(\Xi_n \hat{f}) : N \in [2^n, 2^{n+1}))^2 \right)^{1/2} \right\|_{\ell^p} = I_p^1 + I_p^2. \end{aligned}$$

We will estimate I_p^1 and I_p^2 separately. First, observe that by (5.9) and (5.10), for any $N \simeq 2^n$ and any $a/q \in \mathcal{U}_{n^l}$ we have

$$(5.11) \quad \begin{aligned} |m_N(\xi) - m_{2^n}(\xi)| & \lesssim q^{-\delta} |\Phi_N(\xi - a/q) - \Phi_{2^n}(\xi - a/q)| + q^{-\delta} 2^{-n/2} \\ & \lesssim q^{-\delta} (\min \{1, |2^{nA}(\xi - a/q)|_\infty, |2^{nA}(\xi - a/q)|_\infty^{-1/d}\} + 2^{-n/2}) \end{aligned}$$

where the last bound follows from (4.1) and (4.2). Thus using (5.11) we get

$$(5.12) \quad |(m_N(\xi) - m_{2^n}(\xi))(\eta(2^{nA+jI}(\xi - a/q)) - \eta(2^{nA+(j+1)I}(\xi - a/q)))| \lesssim (2^{-|j|/d} + 2^{-n/2}).$$

We begin with bounding I_p^2 . Since r -variations are decreasing we can assume that $1 < r \leq \min\{2, p\}$ and it will suffice to show, for some $\varepsilon = \varepsilon_{p,r} > 0$, that

$$(5.13) \quad \|V_r((M_N - M_{2^n})\mathcal{F}^{-1}(\Xi_n \hat{f}) : N \in [2^n, 2^{n+1}))\|_{\ell^p} \lesssim 2^{-\varepsilon n} \|f\|_{\ell^p}.$$

Likewise above we shall exploit (2.8) which immediately gives

$$\|V_r((M_N - M_{2^n})\mathcal{F}^{-1}(\Xi_n \hat{f}) : N \in [2^n, 2^{n+1}))\|_{\ell^p} \lesssim \max \{ \mathbf{U}_p, 2^{n/r} \mathbf{U}_p^{1-1/r} \mathbf{V}_p^{1/r} \}$$

where

$$\mathbf{U}_p = \sup_{2^n \leq N \leq 2^{n+1}} \|(M_N - M_{2^n})\mathcal{F}^{-1}(\Xi_n \hat{f})\|_{\ell^p}$$

and

$$\mathbf{V}_p = \sup_{2^n \leq N < 2^{n+1}} \|(M_{N+1} - M_N)\mathcal{F}^{-1}(\Xi_n \hat{f})\|_{\ell^p}.$$

In view of Theorem 3.2 we see that

$$(5.14) \quad \mathbf{U}_p \lesssim \log(n+2) \|f\|_{\ell^p} \quad \text{and} \quad \mathbf{V}_p \lesssim 2^{-n} \log(n+2) \|f\|_{\ell^p}.$$

For $p = 2$ by Plancherel's theorem and (5.11) we obtain

$$(5.15) \quad \begin{aligned} & \| (M_N - M_{2^n}) \mathcal{F}^{-1}(\Xi_n^n \hat{f}) \|_{\ell^2} \\ &= \left(\int_{\mathbb{T}^d} \sum_{a/q \in \mathcal{U}_n^l} |m_N(\xi) - m_{2^n}(\xi)|^2 \eta(2^{nA+nI}(\xi - a/q))^2 |\hat{f}(\xi)|^2 d\xi \right)^{1/2} \lesssim 2^{-n/(2d)} \|f\|_{\ell^2}. \end{aligned}$$

Therefore, interpolating (5.14) with (5.15) we obtain for every $p \in (1, \infty)$ that

$$\mathbf{U}_p \lesssim 2^{-\varepsilon n} \|f\|_{\ell^p}$$

which in turn implies (5.13) and $I_p^2 \lesssim \|f\|_{\ell^p}$.

We shall now estimate I_p^1 , for this purpose we need to define new multipliers for any $0 \leq s < n$

$$\Delta_{n,s}^j(\xi) = \sum_{a/q \in \mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_s^l} (\eta(2^{nA+jI}(\xi - a/q)) - \eta(2^{nA+(j+1)I}(\xi - a/q))) \eta(2^{s(A-\chi I)}(\xi - a/q)),$$

It makes sense since $\mathcal{U}_s^l \subseteq \mathcal{U}_{(s+1)^l}$, thus

$$\Xi_n^j(\xi) - \Xi_n^{j+1}(\xi) = \sum_{0 \leq s < n} \Delta_{n,s}^j(\xi).$$

Now we have

$$\begin{aligned} I_p^1 &= \left\| \left(\sum_{n \geq 0} V_2((M_N - M_{2^n}) \mathcal{F}^{-1} \left(\sum_{-\chi n \leq j < n} \sum_{0 \leq s < n} \Delta_{n,s}^j \hat{f} \right) : N \in [2^n, 2^{n+1}))^2 \right)^{1/2} \right\|_{\ell^p} \\ &\leq \sum_{s \geq 0} \sum_{j \in \mathbb{Z}} \left\| \left(\sum_{n \geq \max\{s, j, -j/\chi\}} V_2((M_N - M_{2^n}) \mathcal{F}^{-1}(\Delta_{n,s}^j \hat{f}) : N \in [2^n, 2^{n+1}))^2 \right)^{1/2} \right\|_{\ell^p}. \end{aligned}$$

The task now is to show that for some $\varepsilon_p > 0$

$$(5.16) \quad J_p = \left\| \left(\sum_{n \geq \max\{s, j, -j/\chi\}} V_2((M_N - M_{2^n}) \mathcal{F}^{-1}(\Delta_{n,s}^j \hat{f}) : N \in [2^n, 2^{n+1}))^2 \right)^{1/2} \right\|_{\ell^p} \lesssim s^{-2} 2^{-\varepsilon_p |j|} \|f\|_{\ell^p}.$$

Before we establish (5.16) we need some prerequisites.

5.2.1. Some preparatory estimates. The next result, more precisely inequality (5.17), can be thought as a discrete counterpart of Littlewood–Paley theory.

Theorem 5.1. *For every $p \in (1, \infty)$ there is a constant $C > 0$ such that for all $f \in \ell^p(\mathbb{Z}^d)$ we have*

$$(5.17) \quad \left\| \left(\sum_{n \geq \max\{s, j, -j/\chi\}} |\mathcal{F}^{-1}(\Delta_{n,s}^j \hat{f})|^2 \right)^{1/2} \right\|_{\ell^p} \leq C \log(s+2) \|f\|_{\ell^p}.$$

Proof. By Khinchine's inequality (5.17) is equivalent to the following

$$(5.18) \quad \left(\int_0^1 \left\| \sum_{n \geq \max\{s, j, -j/\chi\}} \varepsilon_n(t) \mathcal{F}^{-1}(\Delta_{n,s}^j \hat{f}) \right\|_{\ell^p}^p dt \right)^{1/p} \lesssim \log(s+2) \|f\|_{\ell^p}.$$

Indeed, the multiplier from (5.18) can be rewritten as follows

$$\sum_{n \geq \max\{s, j, -j/\chi\}} \varepsilon_j(t) \Delta_{n,s}^j(\xi) = \left(\sum_{a/q \in \mathcal{U}_{(s+1)^k}} - \sum_{a/q \in \mathcal{U}_{s^k}} \right) \sum_{n \geq \max\{s, j, -j/\chi\}} \mathbf{m}_n(\xi - a/q) \eta(2^{s(A-\chi I)}(\xi - a/q))$$

with the functions

$$\mathbf{m}_n(\xi) = \varepsilon_j(t) (\eta(2^{nA+jI}\xi) - \eta(2^{nA+(j+1)A}\xi)).$$

We observe that

$$|\mathbf{m}_n(\xi)| \lesssim \min \{ |2^{nA+jI}\xi|_\infty, |2^{nA+jI}\xi|_\infty^{-1} \}.$$

The first bound follows from the mean-value theorem, since

$$|\eta(2^{nA+jI}\xi) - \eta(2^{nA+(j+1)I}\xi)| \lesssim |2^{nA+jI}\xi - 2^{nA+(j+1)I}\xi| \sup_{\xi \in [-1, 1]^d} |\nabla \eta(\xi)| \lesssim |2^{nA+jI}\xi|_\infty.$$

The second bound follows since η is a Schwartz function. Moreover, for every $p \in (1, \infty)$ we have

$$\left\| \sup_{n \in \mathbb{N}_0} |\mathcal{F}^{-1}(\mathfrak{m}_n \mathcal{F}f)| \right\|_{L^p} \lesssim \|f\|_{L^p}$$

for every $f \in L^p(\mathbb{R}^d)$. Therefore, by [19] the multiplier

$$\sum_{n \geq \max\{s, j, -j/\chi\}} \mathfrak{m}_n(\xi)$$

defines a bounded operator on $L^p(\mathbb{R}^d)$ for all $p \in (1, \infty)$. Hence, Theorem 3.2 applies and one can see that the multiplier

$$\sum_{n \geq \max\{s, j, -j/\chi\}} \varepsilon_n(t) \Delta_{n,s}^j(\xi)$$

defines a bounded operator on $\ell^p(\mathbb{Z}^d)$ with the logarithmic loss with respect to s , and (5.18) is established. \square

To estimate (5.16) we will use (2.6) from Lemma 2.2 with $r = 2$. Namely, let $h_{j,s} = 2^{\varepsilon|j|}(s+1)^\tau$ with some $\varepsilon > 0$ and $\tau > 2$ which we choose later. Then for some $2^n \leq t_0 < t_1 < \dots < t_h < 2^{n+1}$ such that $t_{v+1} - t_v \simeq 2^n/h$ where $h = \min\{h_{j,s}, 2^n\}$ we have

$$\begin{aligned} J_p &\lesssim \left\| \left(\sum_{n \geq \max\{s, j, -j/\chi\}} \sum_{v=0}^h |(M_{2^{t_v}} - M_{2^n})\mathcal{F}^{-1}(\Delta_{n,s}^j \hat{f})|^2 \right)^{1/2} \right\|_{\ell^p} \\ &\quad + \left\| \left(\sum_{n \geq \max\{s, j, -j/\chi\}} \sum_{v=0}^{h-1} \left(\sum_{u=t_v}^{t_{v+1}-1} |(M_{u+1} - M_u)\mathcal{F}^{-1}(\Delta_{n,s}^j \hat{f})|^2 \right)^{1/2} \right) \right\|_{\ell^p} = J_p^1 + J_p^2. \end{aligned}$$

5.2.2. *The estimates for J_p^1 .* We begin with $p = 2$ and show that

$$(5.19) \quad J_2^1 \lesssim 2^{-|j|(1/(2d)-\varepsilon/2)}(s+1)^{-\delta l + \tau/2} \|f\|_{\ell^2}.$$

For the simplicity of notation define

$$\varrho_{n,j}(\xi) = (\eta(2^{nA+jI}\xi) - \eta(2^{nA+(j+1)I}\xi))\eta(2^{s(A-\chi I)}\xi).$$

By Plancherel's theorem and (5.12) we have

$$\begin{aligned} J_2^1 &= \left(\sum_{v=0}^h \int_{\mathbb{T}^d} \sum_{a/q \in \mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_{s^l}} \sum_{n \geq \max\{s, j, -j/\chi\}} |m_{2^{t_v}}(\xi) - m_{2^n}(\xi)|^2 \varrho_{n,j}(\xi - a/q)^2 |\hat{f}(\xi)|^2 d\xi \right)^{1/2} \\ &\lesssim \left(\sum_{v=0}^h 2^{-|j|/d} \int_{\mathbb{T}^d} \sum_{a/q \in \mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_{s^l}} q^{-2\delta} \eta(2^{s(A-\chi I)}(\xi - a/q)) |\hat{f}(\xi)|^2 d\xi \right)^{1/2} \\ &\lesssim h^{1/2} 2^{-|j|/(2d)} \left(\int_{\mathbb{T}^d} (s+1)^{-2\delta l} \sum_{a/q \in \mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_{s^l}} \eta(2^{s(A-\chi I)}(\xi - a/q)) |\hat{f}(\xi)|^2 d\xi \right)^{1/2} \\ &\lesssim h^{1/2} 2^{-|j|/(2d)} (s+1)^{-\delta l} \|f\|_{\ell^2} \end{aligned}$$

as desired. We have used the fact that $q \gtrsim (s+1)^l$ whenever $a/q \in \mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_{s^l}$ and

$$\sum_{n \geq \max\{s, j, -j/\chi\}} \varrho_{n,j}(\xi) \lesssim \eta(2^{s(A-\chi I)}(\xi - a/q))$$

and the disjointness of $\eta(2^{s(A-\chi I)}(\cdot - a/q))$ while a/q varies over $\mathcal{U}_{(s+1)^k} \setminus \mathcal{U}_{s^k}$.

Moreover, for any $p \in (1, \infty)$ we have

$$(5.20) \quad J_p^1 \lesssim 2^{\varepsilon|j|/2} (s+1)^{\tau/2} \log(s+2) \|f\|_{\ell^p}.$$

Indeed, appealing to the vector-valued inequality for the maximal function corresponding to the averaging operators from [15] we see that

$$\begin{aligned} J_p^1 &= \left\| \left(\sum_{n \geq \max\{s, j, -j/\chi\}} \sum_{v=0}^h |(M_{2^{t_v}} - M_{2^n}) \mathcal{F}^{-1}(\Delta_{n,s}^j \hat{f})|^2 \right)^{1/2} \right\|_{\ell^p} \\ &\lesssim h^{1/2} \left\| \left(\sum_{n \geq \max\{s, j, -j/\chi\}} \sup_{N \in \mathbb{N}} M_N(|\mathcal{F}^{-1}(\Delta_{n,s}^j \hat{f})|)^2 \right)^{1/2} \right\|_{\ell^p} \\ &\lesssim h^{1/2} \left\| \left(\sum_{n \geq \max\{s, j, -j/\chi\}} |\mathcal{F}^{-1}(\Delta_{n,s}^j \hat{f})|^2 \right)^{1/2} \right\|_{\ell^p} \lesssim h^{1/2} \log(s+2) \|f\|_{\ell^p}. \end{aligned}$$

In the last step we have used (5.17). Interpolating now (5.20) with better (5.19) estimate, we obtain for some $\varepsilon_p > 0$ that

$$J_p^1 \lesssim (s+1)^{-2} 2^{-\varepsilon_p |j|} \|f\|_{\ell^p}.$$

The proof of (5.16) will be completed if we obtain the same kind of bound for J_p^2 .

5.2.3. *The estimates for J_p^2 .* We begin with $p = 2$ and our aim will be to show

$$(5.21) \quad J_2^2 \lesssim 2^{-\varepsilon |j|/2} (s+1)^{-\tau/2} \|f\|_{\ell^2}.$$

Since $t_{v+1} - t_v \simeq 2^n/h$ then by the Cauchy–Schwarz inequality we obtain

$$J_2^2 \leq \left\| \left(\sum_{n \geq \max\{s, j, -j/\chi\}} 2^n/h \sum_{u=2^n}^{2^{n+1}-1} |(M_{u+1} - M_u) \mathcal{F}^{-1}(\Delta_{n,s}^j \hat{f})|^2 \right)^{1/2} \right\|_{\ell^2}.$$

By (5.12) we have for $u \simeq 2^n$ that

$$(5.22) \quad |m_{u+1}(\xi) - m_u(\xi)| \lesssim \min \{2^{-n}, q^{-\delta} (2^{-|j|/d} + 2^{-n/2})\}.$$

Two cases must be distinguished. Assume now that $h = 2^{\varepsilon |j|} (s+1)^\tau$, therefore, again by Plancherel's theorem, we obtain

$$\begin{aligned} J_2^2 &\leq \left(\sum_{n \geq \max\{s, j, -j/\chi\}} \int_{\mathbb{T}^d} 2^n/h \sum_{u=2^n}^{2^{n+1}-1} \sum_{a/q \in \mathcal{U}_{(s+1)^k} \setminus \mathcal{U}_{s^k}} |m_{u+1}(\xi) - m_u(\xi)|^2 \varrho_{n,j}(\xi - a/q)^2 |\hat{f}(\xi)|^2 d\xi \right)^{1/2} \\ &\lesssim h^{-1/2} \left(\int_{\mathbb{T}^d} \sum_{a/q \in \mathcal{U}_{(s+1)^k} \setminus \mathcal{U}_{s^k}} \sum_{n \geq \max\{s, j, -j/\chi\}} \varrho_{n,j}(\xi - a/q) |\hat{f}(\xi)|^2 d\xi \right)^{1/2} \lesssim h^{-1/2} \|f\|_{\ell^2} \end{aligned}$$

since by the telescoping nature and the disjointness of supports when a/q varies over $\mathcal{U}_{(s+1)^k} \setminus \mathcal{U}_{s^k}$ we have

$$\sum_{a/q \in \mathcal{U}_{(s+1)^k} \setminus \mathcal{U}_{s^k}} \sum_{n \geq \max\{s, j, -j/\chi\}} \varrho_{n,j}(\xi - a/q) \lesssim \sum_{a/q \in \mathcal{U}_{(s+1)^k} \setminus \mathcal{U}_{s^k}} \eta(2^{s(A-\chi I)}(\xi - a/q)) \lesssim 1.$$

If $h = 2^n$ then by (5.22) we get

$$\begin{aligned} J_2^2 &\leq \left(\sum_{n \geq \max\{s, j, -j/\chi\}} \int_{\mathbb{T}^d} \sum_{u=2^n}^{2^{n+1}-1} \sum_{a/q \in \mathcal{U}_{(s+1)^k} \setminus \mathcal{U}_{s^k}} |m_{u+1}(\xi) - m_u(\xi)|^2 \varrho_{n,j}(\xi - a/q)^2 |\hat{f}(\xi)|^2 d\xi \right)^{1/2} \\ &\lesssim 2^{-|j|/4} 2^{-s/4} \left(\int_{\mathbb{T}^d} \sum_{a/q \in \mathcal{U}_{(s+1)^k} \setminus \mathcal{U}_{s^k}} \sum_{n \geq \max\{s, j, -j/\chi\}} \varrho_{n,j}(\xi - a/q) |\hat{f}(\xi)|^2 d\xi \right)^{1/2} \\ &\lesssim 2^{-|j|/4} 2^{-s/4} \|f\|_{\ell^2} \end{aligned}$$

and (5.21) is proven.

For $p \in (1, \infty)$ we shall prove that

$$(5.23) \quad J_p^2 \lesssim \log(s+2) \|f\|_{\ell^p}.$$

Observe that

$$\begin{aligned} \sum_{u=2^n}^{2^{n+1}-1} |K_{u+1} - K_u| &\lesssim \sum_{u=2^n}^{2^{n+1}-1} \left(\frac{1}{(u+1)^k} \sum_{y \in \mathbb{B}_{u+1} \setminus \mathbb{B}_u} \delta_{\mathcal{Q}(y)} + \frac{1}{u(u+1)^k} \sum_{y \in \mathbb{B}_u} \delta_{\mathcal{Q}(y)} \right) \\ &\lesssim \frac{1}{2^{nk}} \sum_{y \in \mathbb{B}_{2^{n+1}} \setminus \mathbb{B}_{2^n}} \delta_{\mathcal{Q}(y)} + \frac{1}{2^{(n+1)k}} \sum_{y \in \mathbb{B}_{2^{n+1}}} \delta_{\mathcal{Q}(y)} \lesssim K_{2^{n+1}}. \end{aligned}$$

This in turn implies

$$\begin{aligned} J_p^2 &= \left\| \left(\sum_{n \geq \max\{s, j, -j/\chi\}} \sum_{v=0}^{h-1} \left(\sum_{u=t_v}^{t_{v+1}-1} |(M_{u+1} - M_u) \mathcal{F}^{-1}(\Delta_{n,s}^j \hat{f})|^2 \right)^{1/2} \right) \right\|_{\ell^p} \\ &\leq \left\| \left(\sum_{n \geq \max\{s, j, -j/\chi\}} \left(\sum_{u=2^n}^{2^{n+1}-1} |(K_{u+1} - K_u)| * (|\mathcal{F}^{-1}(\Delta_{n,s}^j \hat{f})|)^2 \right)^{1/2} \right) \right\|_{\ell^p} \\ &\lesssim \left\| \left(\sum_{n \geq \max\{s, j, -j/\chi\}} \sup_{N \in \mathbb{N}} M_N(|\mathcal{F}^{-1}(\Delta_{n,s}^j \hat{f})|^2) \right)^{1/2} \right\|_{\ell^p} \\ &\lesssim \left\| \left(\sum_{n \geq \max\{s, j, -j/\chi\}} |\mathcal{F}^{-1}(\Delta_{n,s}^j \hat{f})|^2 \right)^{1/2} \right\|_{\ell^p} \lesssim \log(s+2) \|f\|_{\ell^p}. \end{aligned}$$

In the penultimate line we have used vector-valued maximal estimates corresponding to the averaging operators from [15] and in the last line we invoked (5.17). Interpolating now the estimate (5.23) with the estimate from (5.21) we obtain for some $\varepsilon_p > 0$ that

$$J_p^2 \lesssim (s+1)^{-2} 2^{-\varepsilon_p |j|} \|f\|_{\ell^p}$$

and the proof of (5.16) is completed.

6. LONG VARIATION ESTIMATES FOR TRUNCATED SINGULAR INTEGRAL OPERATORS

For any function $f : \mathbb{Z}^d \rightarrow \mathbb{C}$ with a finite support we have

$$T_N f(x) = H_N * f(x)$$

with a kernel H_N defined by

$$H_N(x) = \sum_{y \in \mathbb{B}_N \setminus \{0\}} \delta_{\mathcal{Q}(y)} K(y)$$

where K is the kernel as in (1.14) and δ_y denotes Dirac's delta at $y \in \mathbb{Z}^k$ and \mathcal{Q} is the canonical polynomial, see Section 2. Let m_N denote the discrete Fourier transform of H_N , i.e.

$$m_N(\xi) = \sum_{y \in \mathbb{B}_N \setminus \{0\}} e^{2\pi i \xi \cdot \mathcal{Q}(y)} K(y).$$

Finally, we define

$$\Psi_t(\xi) = \text{p.v.} \int_{B_t} e^{2\pi i \xi \cdot \mathcal{Q}(y)} K(y) \, dy$$

where B_t is the Euclidean ball in \mathbb{R}^k centered at the origin with radius $t > 0$. Using the method of the proof of the multi-dimensional version of van der Corput lemma in [21] we may estimate

$$(6.1) \quad |\Psi_N(\xi) - \Psi_{cN}(\xi)| = \left| \int_{B_1 \setminus B_c} e^{2\pi i \xi \cdot \mathcal{Q}(Ny)} N^k K(Ny) \, dy \right| \lesssim \min \{1, |N^A \xi|_\infty^{-1/d}\}$$

with the implicit constant depending on $c \in (0, 1)$. Additionally, we have

$$(6.2) \quad |\Psi_N(\xi) - \Psi_{cN}(\xi)| \lesssim |N^A \xi|_\infty$$

due to cancellation condition (1.2). We shall prove that for every $p \in (1, \infty)$ and $r \in (2, \infty)$ there is $C_{p,r} > 0$ such that for all $f \in \ell^p(\mathbb{Z}^{d_0})$ and $f \geq 0$ we have

$$\|V_r(T_{2^n} f : n \in \mathbb{N}_0)\|_{\ell^p} \leq C_{p,r} \|f\|_{\ell^p}$$

and $C_{p,r} \leq C_p \frac{r}{r-2}$ for some $C_p > 0$. We begin with proving the following, which is a variant of Proposition 4.1.

Proposition 6.1. *There is a constant $C > 0$ such that for every $N \in \mathbb{N}$, $M \in \mathbb{N}$ such that $cN \leq M \leq N$ for some $c > 0$ and for every $\xi \in [-1/2, 1/2]^d$ satisfying*

$$\left| \xi_\gamma - \frac{a_\gamma}{q} \right| \leq L_1^{-|\gamma|} L_2$$

for all $\gamma \in \Gamma$, where $1 \leq q \leq L_3 \leq N^{1/2}$, $a \in A_q$, $L_1 \geq N$ and $L_2 \geq 1$ we have

$$|m_N(\xi) - m_M(\xi) - G(a/q)(\Psi_N(\xi - a/q) - \Psi_M(\xi - a/q))| \leq C \left(L_3/N + L_2 L_3/N \sum_{\gamma \in \Gamma} (N/L_1)^{|\gamma|} \right).$$

Proof. Let $\theta = \xi - a/q$. For any $r \in \mathbb{N}_q^k$, if $y \equiv r \pmod{q}$ then for each $\gamma \in \Gamma$

$$\xi_\gamma y^\gamma \equiv \theta_\gamma y^\gamma + (a_\gamma/q) r^\gamma \pmod{1},$$

thus

$$\xi \cdot \mathcal{Q}(y) \equiv \theta \cdot \mathcal{Q}(y) + (a/q) \cdot \mathcal{Q}(r) \pmod{1}.$$

Therefore,

$$\sum_{y \in \mathbb{B}_N \setminus \mathbb{B}_M} e^{2\pi i \xi \cdot \mathcal{Q}(y)} K(y) = q^{-k} \sum_{r \in \mathbb{N}_q^k} e^{2\pi i (a/q) \cdot \mathcal{Q}(r)} \cdot \left(q^k \sum_{\substack{y \in \mathbb{Z}^k \\ qy+r \in \mathbb{B}_N \setminus \mathbb{B}_M}} e^{2\pi i \theta \cdot \mathcal{Q}(qy+r)} K(qy+r) \right).$$

If $qy + r \in \mathbb{B}_N \setminus \mathbb{B}_M$ then

$$|\theta \cdot \mathcal{Q}(qy+r) - \theta \cdot \mathcal{Q}(qy)| \lesssim |r| \sum_{\gamma \in \Gamma} |\theta_\gamma| \cdot N^{(|\gamma|-1)} \lesssim q \sum_{\gamma \in \Gamma} L_1^{-|\gamma|} L_2 N^{(|\gamma|-1)} \lesssim L_2 L_3/N \sum_{\gamma \in \Gamma} (N/L_1)^{|\gamma|}$$

and

$$|K(qy+r) - K(qy)| \lesssim N^{-(k+1)} q.$$

Thus

$$\begin{aligned} q^k \sum_{\substack{y \in \mathbb{Z}^k \\ qy+r \in \mathbb{B}_N \setminus \mathbb{B}_M}} e^{2\pi i \theta \cdot \mathcal{Q}(qy+r)} K(qy+r) &= q^k \sum_{\substack{y \in \mathbb{Z}^k \\ qy+r \in \mathbb{B}_N \setminus \mathbb{B}_M}} e^{2\pi i \theta \cdot \mathcal{Q}(qy)} K(qy) \\ &\quad + \mathcal{O}\left(q/N + L_2 L_3/N \sum_{\gamma \in \Gamma} (N/L_1)^{|\gamma|}\right). \end{aligned}$$

Now we see that

$$q^k \sum_{\substack{y \in \mathbb{Z}^k \\ qy+r \in \mathbb{B}_N \setminus \mathbb{B}_M}} e^{2\pi i \theta \cdot \mathcal{Q}(qy)} K(qy) = q^k \sum_{\substack{y \in \mathbb{Z}^k \\ qy \in \mathbb{B}_N \setminus \mathbb{B}_M}} e^{2\pi i \theta \cdot \mathcal{Q}(qy)} K(qy) + \mathcal{O}(q/N).$$

Thus

$$\sum_{y \in \mathbb{B}_N \setminus \mathbb{B}_M} e^{2\pi i \xi \cdot \mathcal{Q}(y)} K(y) = G(a/q) \cdot q^k \sum_{\substack{y \in \mathbb{Z}^k \\ qy \in \mathbb{B}_N \setminus \mathbb{B}_M}} e^{2\pi i \theta \cdot \mathcal{Q}(qy)} K(qy) + \mathcal{O}\left(q/N + L_2 L_3/N \sum_{\gamma \in \Gamma} (N/L_1)^{|\gamma|}\right).$$

Now we are going to replace the exponential sum on the right-hand side of the last display by the integral. By the mean value theorem, we obtain

$$\begin{aligned} q^k \left| \sum_{\substack{y \in \mathbb{Z}^k \\ qy \in \mathbb{B}_N \setminus \mathbb{B}_M}} e^{2\pi i \theta \cdot \mathcal{Q}(qy)} K(qy) - \int_{B_{N/q} \setminus B_{M/q}} e^{2\pi i \theta \cdot \mathcal{Q}(qt)} K(qt) dt \right| \\ = q^k \left| \sum_{y \in \mathbb{Z}^k} e^{2\pi i \theta \cdot \mathcal{Q}(qy)} K(qy) \mathbb{1}_{\mathbb{B}_N \setminus \mathbb{B}_M}(qy) - \sum_{y \in \mathbb{Z}^k} \int_{y+(0,1]^k} e^{2\pi i \theta \cdot \mathcal{Q}(qt)} K(qt) \mathbb{1}_{B_{N/q} \setminus B_{M/q}}(t) dt \right| \\ = q^k \left| \sum_{y \in \mathbb{Z}^k} \int_{(0,1]^k} e^{2\pi i \theta \cdot \mathcal{Q}(qy)} K(qy) \mathbb{1}_{\mathbb{B}_N \setminus \mathbb{B}_M}(qy) - e^{2\pi i \theta \cdot \mathcal{Q}(q(t+y))} K(q(t+y)) \mathbb{1}_{\mathbb{B}_N \setminus \mathbb{B}_M}(q(y+t)) dt \right| \\ = \mathcal{O}\left(q/N + L_2 L_3/N \sum_{\gamma \in \Gamma} (N/L_1)^{|\gamma|}\right). \end{aligned}$$

This completes the proof of Proposition 6.1. \square

In Remark 1.2 we mentioned that Theorem B holds with the operators T_N defined with the sets \mathbb{G}_N instead of \mathbb{B}_N . Then we obtain analogous definitions of H_N , m_N and Ψ_N with the sets \mathbb{G}_N , $G_1 = G$ and G_t in place of the sets \mathbb{B}_N , B_1 and B_t respectively. All of the arguments remain unchanged apart from the proof of Proposition 6.1. However, [15, Proposition 3.1] used with the sets \mathbb{G}_N allows us to follow the same scheme and we obtain conclusion of the same type.

As in the previous sections fix the numbers $\chi > 0$ and $l \in \mathbb{N}$ whose precise values will be chosen later, and let us consider for every $n \in \mathbb{N}_0$ the multipliers

$$(6.3) \quad \Xi_n(\xi) = \sum_{a/q \in \mathcal{U}_{n^l}} \eta(2^{n(A-\chi l)}(\xi - a/q))$$

with \mathcal{U}_{n^l} defined as in (3.4). Theorem 3.2 guarantees that for every $p \in (1, \infty)$

$$(6.4) \quad \|\mathcal{F}^{-1}(\Xi_n \hat{f})\|_{\ell^p} \lesssim \log(n+2) \|f\|_{\ell^p}.$$

The implicit constant in (6.4) depends on the parameter $\rho > 0$, see Section 3. However, from now on we will assume that $\rho > 0$ and the integer $l \geq 10$ are related by the equation

$$(6.5) \quad 10\rho l = 1.$$

Observe that

$$(6.6) \quad \|V_r(T_{2^n} f : n \in \mathbb{N}_0)\|_{\ell^p} \leq \|V_r\left(\mathcal{F}^{-1}\left(\sum_{j=1}^n (m_{2^j} - m_{2^{j-1}}) \Xi_j \hat{f}\right) : n \in \mathbb{N}_0\right)\|_{\ell^p} \\ + \|V_r\left(\mathcal{F}^{-1}\left(\sum_{j=1}^n (m_{2^j} - m_{2^{j-1}})(1 - \Xi_j) \hat{f}\right) : n \in \mathbb{N}_0\right)\|_{\ell^p}.$$

6.1. The estimate for the second norm in (6.6). Since the variational norm is increasing when r decreases we get

$$\|V_1\left(\mathcal{F}^{-1}\left(\sum_{j=1}^n (m_{2^j} - m_{2^{j-1}})(1 - \Xi_j) \hat{f}\right) : n \in \mathbb{N}_0\right)\|_{\ell^p} \leq \sum_{n \in \mathbb{N}_0} \|\mathcal{F}^{-1}((m_{2^n} - m_{2^{n-1}})(1 - \Xi_n) \hat{f})\|_{\ell^p}.$$

Therefore, it suffices to show that

$$(6.7) \quad \|\mathcal{F}^{-1}((m_{2^n} - m_{2^{n-1}})(1 - \Xi_n) \hat{f})\|_{\ell^p} \leq (n+1)^{-2} \|f\|_{\ell^p}.$$

For every $1 < p < \infty$ we have

$$(6.8) \quad \|\mathcal{F}^{-1}((m_{2^n} - m_{2^{n-1}})(1 - \Xi_n) \hat{f})\|_{\ell^p} \lesssim \|M_{2^n} f\|_{\ell^p} + \|M_{2^n}(\mathcal{F}^{-1}(\Xi_n \hat{f}))\|_{\ell^p} \lesssim \log(n+2) \|f\|_{\ell^p}$$

since for $f \geq 0$ we have a pointwise bound

$$|\mathcal{F}^{-1}((m_{2^n} - m_{2^{n-1}}) \hat{f})(x)| \lesssim M_{2^n} f(x)$$

where M_N is the averaging operator from Section 4. In fact we improve estimate (6.8) for $p = 2$. Indeed, we will show that for big enough $\alpha > 0$, which will be specified later, and for all $n \in \mathbb{N}_0$ we have

$$(6.9) \quad |(m_{2^n}(\xi) - m_{2^{n-1}}(\xi))(1 - \Xi_n(\xi))| \lesssim (n+1)^{-\alpha}.$$

This estimate will be a consequence of Theorem 3.1. For do so, by Dirichlet's principle we have for every $\gamma \in \Gamma$

$$\left| \xi_\gamma - \frac{a_\gamma}{q_\gamma} \right| \leq \frac{n^\beta}{q_\gamma 2^{n|\gamma|}}$$

where $1 \leq q_\gamma \leq n^{-\beta} 2^{n|\gamma|}$. In order to apply Theorem 3.1 we must show that there exists some $\gamma \in \Gamma$ such that $n^\beta \leq q_\gamma \leq n^{-\beta} 2^{n|\gamma|}$. Suppose for a contradiction that for every $\gamma \in \Gamma$ we have $1 \leq q_\gamma < n^\beta$ then for some $q \leq \text{lcm}(q_\gamma : \gamma \in \Gamma) \leq n^{\beta d}$ we have

$$\left| \xi_\gamma - \frac{a'_\gamma}{q} \right| \leq \frac{n^\beta}{2^{n|\gamma|}}$$

where $\gcd(q, \gcd(a'_\gamma : \gamma \in \Gamma)) = 1$. Hence, taking $a' = (a'_\gamma : \gamma \in \Gamma)$ we have $a'/q \in \mathcal{U}_{n^l}$ provided that $\beta d < l$. On the other hand, if $1 - \Xi_n(\xi) \neq 0$ then for every $a'/q \in \mathcal{U}_{n^l}$ there exists $\gamma \in \Gamma$ such that

$$\left| \xi_\gamma - \frac{a'_\gamma}{q} \right| > \frac{1}{16d \cdot 2^{n(|\gamma| - \chi)}}.$$

Therefore, one obtains

$$2^{\chi n} < 16dn^\beta$$

but this gives a contradiction, for sufficiently large $n \in \mathbb{N}$. We have already shown that there exists some $\gamma \in \Gamma$ such that $n^\beta \leq q_\gamma \leq n^{-\beta} 2^{n|\gamma|}$ and consequently Theorem 3.1 yields

$$|m_{2^n}(\xi) - m_{2^{n-1}}(\xi)| \lesssim (n+1)^{-\alpha}$$

provided that $1 - \Xi_n(\xi) \neq 0$ and this proves (6.9) and we obtain

$$(6.10) \quad \|\mathcal{F}^{-1}((m_{2^n} - m_{2^{n-1}})(1 - \Xi_n)\hat{f})\|_{\ell^2} \lesssim (1+n)^{-\alpha} \log(n+2) \|f\|_{\ell^2}.$$

Interpolating (6.10) with (6.8) we obtain

$$\|\mathcal{F}^{-1}((m_{2^n} - m_{2^{n-1}})(1 - \Xi_n)\hat{f})\|_{\ell^p} \lesssim (1+n)^{-c_p \alpha} \log(n+2) \|f\|_{\ell^p}.$$

for some $c_p > 0$. Choosing $\alpha > 0$ and $l \in \mathbb{N}$ appropriately large one obtains (6.7).

6.2. The estimate for the first norm in (6.6). Note that for any $\xi \in \mathbb{T}^d$ so that

$$\left| \xi_\gamma - \frac{a_\gamma}{q} \right| \leq \frac{1}{8d \cdot 2^{j(|\gamma| - \chi)}}$$

for every $\gamma \in \Gamma$ with $1 \leq q \leq e^{j^{1/10}}$ we have

$$(6.11) \quad m_{2^j}(\xi) - m_{2^{j-1}}(\xi) = G(a/q)(\Psi_{2^j}(\xi - a/q) - \Psi_{2^{j-1}}(\xi - a/q)) + q^{-\delta} E_{2^j}(\xi)$$

where

$$(6.12) \quad |E_{2^j}(\xi)| \lesssim 2^{-j/2}.$$

These two properties (6.11) and (6.12) follow from Proposition 6.1 with $L_1 = 2^j$, $L_2 = 8d \cdot 2^{\chi j}$ and $L_3 = e^{j^{1/10}}$, since

$$|E_{2^j}(\xi)| \lesssim q^\delta L_2 L_3 2^{-j} \lesssim (e^{-j((1-\chi) \log 2 - 2j^{-9/10})}) \lesssim 2^{-j/2}$$

which holds for sufficiently large $j \in \mathbb{N}$, when $\chi > 0$ is sufficiently small. Let us introduce for every $j \in \mathbb{N}$ new multipliers

$$\nu_{2^j}(\xi) = \sum_{a/q \in \mathcal{U}_{j^l}} G(a/q)(\Psi_{2^j}(\xi - a/q) - \Psi_{2^{j-1}}(\xi - a/q)) \eta(2^{j(A-\chi I)}(\xi - a/q))$$

and note that by (6.11)

$$|(m_{2^j}(\xi) - m_{2^{j-1}}(\xi))\Xi_j(\xi) - \nu_{2^j}(\xi)| \lesssim 2^{-j/2}$$

and consequently by Plancherel's theorem

$$(6.13) \quad \|\mathcal{F}^{-1}((m_{2^j} - m_{2^{j-1}})\Xi_j - \nu_{2^j})\hat{f}\|_{\ell^2} \lesssim 2^{-j/2} \|f\|_{\ell^2}.$$

Moreover, by Theorem 3.2 we have

$$\|\mathcal{F}^{-1}((m_{2^j} - m_{2^{j-1}})\Xi_j \hat{f})\|_{\ell^p} \lesssim \log(j+2) \|f\|_{\ell^p}$$

and

$$\|\mathcal{F}^{-1}(\nu_{2^j} \hat{f})\|_{\ell^p} \lesssim |\mathcal{U}_{j^l}| \|f\|_{\ell^p} \lesssim e^{(d+1)j^{1/10}} \|f\|_{\ell^p}$$

thus

$$(6.14) \quad \|\mathcal{F}^{-1}((m_{2^j} - m_{2^{j-1}})\Xi_j - \nu_{2^j})\hat{f}\|_{\ell^p} \lesssim e^{(d+1)j^{1/10}} \|f\|_{\ell^p}.$$

Interpolating now (6.13) with (6.14) we can conclude that for some $c_p > 0$

$$(6.15) \quad \|\mathcal{F}^{-1}((m_{2^j} - m_{2^{j-1}})\Xi_j - \nu_{2^j})\hat{f}\|_{\ell^p} \lesssim 2^{-c_p j} \|f\|_{\ell^p}.$$

For every $j \in \mathbb{N}$, $s \in \mathbb{N}_0$ define multipliers

$$\nu_{2^j}^s(\xi) = \sum_{a/q \in \mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_{s^l}} G(a/q)(\Psi_{2^j}(\xi - a/q) - \Psi_{2^{j-1}}(\xi - a/q)) \eta(2^{s(A-\chi I)}(\xi - a/q))$$

and note that by (6.1) we see

$$(6.16) \quad \left| \nu_{2^j}(\xi) - \sum_{0 \leq s < j} \nu_{2^s}(\xi) \right| \leq \sum_{0 \leq s < j} \sum_{a/q \in \mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_s} |G(a/q)| |\Psi_{2^j}(\xi - a/q) - \Psi_{2^{j-1}}(\xi - a/q)| |\eta(2^{s(A-\chi I)}(\xi - a/q)) - \eta(2^{j(A-\chi I)}(\xi - a/q))| \lesssim 2^{-\chi j/d}$$

since $|\Psi_{2^j}(\xi - a/q) - \Psi_{2^{j-1}}(\xi - a/q)| \lesssim 2^{-\chi j/d}$, provided that $\eta(2^{s(A-\chi I)}(\xi - a/q)) - \eta(2^{j(A-\chi I)}(\xi - a/q)) \neq 0$. The estimate (6.16) combined with Plancherel's theorem implies that

$$(6.17) \quad \left\| \mathcal{F}^{-1} \left(\left(\nu_{2^j} - \sum_{0 \leq s < j} \nu_{2^s} \right) \hat{f} \right) \right\|_{\ell^2} \lesssim 2^{-\chi j/d} \|f\|_{\ell^2}$$

Moreover, since $|\mathcal{U}_{s^l}| \leq |\mathcal{U}_{j^l}| \lesssim e^{(d+1)j^{1/10}}$ we have

$$(6.18) \quad \left\| \mathcal{F}^{-1} \left(\left(\nu_{2^j} - \sum_{0 \leq s < j} \nu_{2^s} \right) \hat{f} \right) \right\|_{\ell^p} \lesssim e^{(d+1)j^{1/10}} \|f\|_{\ell^p}.$$

Interpolating (6.17) with (6.18) one immediately concludes that for some $c_p > 0$

$$(6.19) \quad \left\| \mathcal{F}^{-1} \left(\left(\nu_{2^j} - \sum_{0 \leq s < j} \nu_{2^s} \right) \hat{f} \right) \right\|_{\ell^p} \lesssim 2^{-c_p j} \|f\|_{\ell^p}.$$

In view of (6.15) and (6.19) it suffices to prove that for every $s \in \mathbb{N}_0$ we have

$$(6.20) \quad \left\| V_r \left(\mathcal{F}^{-1} \left(\sum_{j=s+1}^n \nu_{2^j} \hat{f} \right) : n \in \mathbb{N}_0 \right) \right\|_{\ell^p} \lesssim (s+1)^{-2} \|f\|_{\ell^p}.$$

6.3. $\ell^2(\mathbb{Z}^d)$ estimates for (6.20). Our aim will be to prove the following.

Theorem 6.1. *For every $r \in (2, \infty)$ there is $C_r > 0$ such that for any $s \in \mathbb{N}_0$ and $f \in \ell^2(\mathbb{Z}^d)$*

$$(6.21) \quad \left\| V_r \left(\mathcal{F}^{-1} \left(\sum_{j=s+1}^n \nu_{2^j} \hat{f} \right) : n \in \mathbb{N}_0 \right) \right\|_{\ell^2} \leq C_r (s+1)^{-\delta l+1} \|f\|_{\ell^2}$$

with $l \in \mathbb{N}_0$ defined as in (6.5) and $\delta > 0$ as in (3.1).

Proof. For $s \in \mathbb{N}_0$ we set $\kappa_s = 20d(\lfloor (s+1)^{1/10} \rfloor + 1)$ and $Q_s = (\lfloor e^{(s+1)^{1/10}} \rfloor)!$. We shall estimate separately the pieces of r -variations where $0 \leq n \leq 2^{\kappa_s}$ and $n \geq 2^{\kappa_s}$. By (2.2) and (2.1) we see that

$$(6.22) \quad \left\| V_r \left(\mathcal{F}^{-1} \left(\sum_{j=s+1}^n \nu_{2^j} \hat{f} \right) : n \in \mathbb{N}_0 \right) \right\|_{\ell^2} \lesssim \left\| \mathcal{F}^{-1}(\nu_{2^{s+1}} \hat{f}) \right\|_{\ell^2} + \left\| V_r \left(\mathcal{F}^{-1} \left(\sum_{j=s+1}^n \nu_{2^j} \hat{f} \right) : 0 \leq n \leq 2^{\kappa_s} \right) \right\|_{\ell^2} + \left\| V_r \left(\mathcal{F}^{-1} \left(\sum_{j=s+1}^n \nu_{2^j} \hat{f} \right) : n \geq 2^{\kappa_s} \right) \right\|_{\ell^2}.$$

By Plancherel's theorem, (3.1) and the disjointness of supports of $\eta_s(\xi - a/q)$'s while a/q varies over $\mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_{s^l}$, the first term in (6.22) is bounded by $(s+1)^{-\delta l} \|f\|_{\ell^2}$. Now we estimate the supremum over $0 \leq n \leq 2^{\kappa_s}$. By Lemma 2.1 we have

$$\left\| V_r \left(\mathcal{F}^{-1} \left(\sum_{j=s+1}^n \nu_{2^j} \hat{f} \right) : 0 \leq n \leq 2^{\kappa_s} \right) \right\|_{\ell^2} \lesssim \sum_{i=0}^{\kappa_s} \left(\sum_{j=0}^{2^{\kappa_s-i}-1} \left\| \sum_{m \in I_j^i} \mathcal{F}^{-1}(\nu_{2^m} \hat{f}) \right\|_{\ell^2}^2 \right)^{1/2}$$

where $I_j^i = (j2^i, (j+1)2^i]$. For any $i \in \{0, \dots, \kappa_s\}$, by Plancherel's theorem we get

$$\begin{aligned} \sum_{j=0}^{2^{\kappa_s-i}-1} \left\| \sum_{m \in I_j^i} \mathcal{F}^{-1}(\nu_{2^m} \hat{f}) \right\|_{\ell^2}^2 &= \sum_{j=0}^{2^{\kappa_s-i}-1} \sum_{m, m' \in I_j^i} \int_{\mathbb{T}^d} |\nu_{2^m}(\xi)| \cdot |\nu_{2^{m'}}(\xi)| |\hat{f}(\xi)|^2 d\xi \\ &\leq \sum_{a/q \in \mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_{s^l}} |G(a/q)|^2 \sum_{j=0}^{2^{\kappa_s-i}-1} \sum_{m, m' \in I_j^i} \int_{\mathbb{T}^d} |\Delta_m(\xi - a/q)| \cdot |\Delta_{m'}(\xi - a/q)| \cdot \eta_s(\xi - a/q)^2 |\hat{f}(\xi)|^2 d\xi, \end{aligned}$$

where $\Delta_m(\xi) = \Psi_{2^m}(\xi) - \Psi_{2^{m-1}}(\xi)$ and $\eta_s(\xi) = \eta(2^{s(A-\chi I)}\xi)$, since the supports are effectively disjoint. Using (6.1) and (6.2) we conclude

$$\sum_{m \in \mathbb{Z}} |\Delta_m(\xi)| \lesssim \sum_{m \in \mathbb{Z}} \min \{ |2^{mA} \xi|_\infty, |2^{mA} \xi|_\infty^{-1/d} \} \lesssim 1.$$

Therefore, by (3.1) we may estimate

$$\sum_{j=0}^{2^{\kappa_s} - i - 1} \left\| \sum_{m \in I_j^i} \mathcal{F}^{-1}(\nu_{2^m}^s \hat{f}) \right\|_{\ell^2}^2 \lesssim (s+1)^{-2\delta l} \sum_{a/q \in \mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_{s^l}} \int_{\mathbb{T}^d} \eta_s(\xi - a/q)^2 |\hat{f}(\xi)|^2 d\xi \lesssim (s+1)^{-2\delta l} \|f\|_{\ell^2}^2$$

since if $a/q \in \mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_{s^l}$ then $q \gtrsim (s+1)^l$. In the last step we have used disjointness of supports of $\eta_s(\cdot - a/q)$ while a/q varies over $\mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_{s^l}$. We have just proven

$$(6.23) \quad \left\| V_r \left(\mathcal{F}^{-1} \left(\sum_{j=s+1}^n \nu_{2^j}^s \hat{f} \right) : 0 \leq n \leq 2^{\kappa_s} \right) \right\|_{\ell^2} \lesssim \kappa_s (s+1)^{-\delta l} \|f\|_{\ell^2} \lesssim (s+1)^{-\delta l + 1} \|f\|_{\ell^2}.$$

Next, we consider the case when the supremum is taken over $n \geq 2^{\kappa_s}$. For any $x, y \in \mathbb{Z}^d$ we define

$$I(x, y) = V_r \left(\sum_{a/q \in \mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_{s^l}} G(a/q) e^{-2\pi i(a/q) \cdot x} \mathcal{F}^{-1} \left(\sum_{j=s+1}^n (\Psi_{2^j} - \Psi_{2^{j-1}}) \eta_s \hat{f}(\cdot + a/q) \right) (y) : n \geq 2^{\kappa_s} \right)$$

and

$$J(x, y) = \sum_{a/q \in \mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_{s^l}} G(a/q) e^{-2\pi i(a/q) \cdot x} \mathcal{F}^{-1}(\eta_s \hat{f}(\cdot + a/q))(y).$$

By Plancherel's theorem, for any $u \in \mathbb{N}_{Q_s}^d$ and $a/q \in \mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_{s^l}$ we have

$$\begin{aligned} & \left\| \mathcal{F}^{-1}((\Psi_{2^n} - \Psi_{2^{n-1}}) \eta_s \hat{f}(\cdot + a/q))(x+u) - \mathcal{F}^{-1}((\Psi_{2^n} - \Psi_{2^{n-1}}) \eta_s \hat{f}(\cdot + a/q))(x) \right\|_{\ell^2(x)} \\ &= \left\| (1 - e^{-2\pi i \xi \cdot u}) (\Psi_{2^n}(\xi) - \Psi_{2^{n-1}}(\xi)) \eta_s(\xi) \hat{f}(\xi + a/q) \right\|_{L^2(d\xi)} \lesssim 2^{-n/d} \cdot |u| \cdot \left\| \eta_s(\cdot - a/q) \hat{f} \right\|_{L^2} \end{aligned}$$

since, by (6.1),

$$\sup_{\xi \in \mathbb{T}^d} |\xi| \cdot |(\Psi_{2^n}(\xi) - \Psi_{2^{n-1}}(\xi))| \lesssim \sup_{\xi \in \mathbb{T}^d} |\xi| \cdot |2^{nA} \xi|^{-1/d} \leq 2^{-n/d}.$$

Therefore,

$$\left| \|I(x, x+u)\|_{\ell^2(x)} - \|I(x, x)\|_{\ell^2(x)} \right| \leq |u| \sum_{n=2^{\kappa_s}}^{\infty} 2^{-n/d} \sum_{a/q \in \mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_{s^l}} \left\| \eta_s(\cdot - a/q) \hat{f} \right\|_{\ell^2}$$

because the set $\mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_{s^l} \subseteq \mathcal{U}_{(s+1)^l}$ contains at most $e^{(d+1)(s+1)^{1/10}}$ elements and

$$2^{\kappa_s} (\log 2)/d - (s+1)^{1/10} e^{(s+1)^{1/10}} - (d+1)(s+1)^{1/10} \geq s$$

for sufficiently large $s \geq 0$. Thus we obtain

$$\|I(x, x)\|_{\ell^2(x)} \lesssim \|I(x, x+u)\|_{\ell^2(x)} + 2^{-s} \|f\|_{\ell^2}.$$

In particular,

$$\left\| V_r \left(\mathcal{F}^{-1} \left(\sum_{j=s+1}^n \nu_{2^j}^s \hat{f} \right) : n \geq 2^{\kappa_s} \right) \right\|_{\ell^2}^2 \lesssim \frac{1}{Q_s^d} \sum_{u \in \mathbb{N}_{Q_s}^d} \|I(x, x+u)\|_{\ell^2(x)}^2 + 2^{-2s} \|f\|_{\ell^2}^2.$$

Let us observe that the functions $x \mapsto I(x, y)$ and $x \mapsto J(x, y)$ are $Q_s \mathbb{Z}^d$ -periodic. Next, by double change of variables and periodicity we get

$$\sum_{u \in \mathbb{N}_{Q_s}^d} \|I(x, x+u)\|_{\ell^2(x)}^2 = \sum_{x \in \mathbb{Z}^d} \sum_{u \in \mathbb{N}_{Q_s}^d} I(x-u, x)^2 = \sum_{x \in \mathbb{Z}^d} \sum_{u \in \mathbb{N}_{Q_s}^d} I(u, x)^2 = \sum_{u \in \mathbb{N}_{Q_s}^d} \|I(u, x)\|_{\ell^2(x)}^2$$

Using Proposition 3.1 and (3.1), we obtain

$$\begin{aligned} \sum_{u \in \mathbb{N}_{Q_s}^d} \|I(u, x)\|_{\ell^2(x)}^2 &\lesssim \sum_{u \in \mathbb{N}_{Q_s}^d} \|J(u, x)\|_{\ell^2(x)}^2 = \sum_{u \in \mathbb{N}_{Q_s}^d} \|J(x, x+u)\|_{\ell^2(x)}^2 \\ &= \sum_{u \in \mathbb{N}_{Q_s}^d} \int_{\mathbb{T}^d} \left| \sum_{a/q \in \mathcal{U}_{(s+1)^t} \setminus \mathcal{U}_s^t} G(a/q) e^{2\pi i(a/q) \cdot u} \eta_s(\xi - a/q) \hat{f}(\xi) \right|^2 d\xi \lesssim (s+1)^{-2\delta t} Q_s^d \cdot \|f\|_{\ell^2}^2. \end{aligned}$$

In the last step we have also used the disjointness of supports of $\eta_s(\cdot - a/q)$ while a/q varies over $\mathcal{U}_{(s+1)^t} \setminus \mathcal{U}_s^t$. Therefore,

$$\left\| V_r \left(\mathcal{F}^{-1} \left(\sum_{j=s+1}^n \nu_{2^j}^s \hat{f} \right) : n \geq 2^{\kappa_s} \right) \right\|_{\ell^2} \lesssim (s+1)^{-\delta t} \|f\|_{\ell^2}$$

which together with (6.23) concludes the proof. \square

6.4. $\ell^p(\mathbb{Z}^d)$ estimates for (6.20). Recall that for $s \in \mathbb{N}_0$ we have

$$\kappa_s = 20d(\lfloor (s+1)^{1/10} \rfloor + 1)$$

and

$$Q_s = (\lfloor e^{(s+1)^{1/10}} \rfloor)!$$

as in the proof of Theorem 6.1. We show that for every $p \in (1, \infty)$ and $r \in (2, \infty)$ there is a constant $C_{p,r} > 0$ such that for every $s \in \mathbb{N}_0$

$$(6.24) \quad \left\| V_r \left(\mathcal{F}^{-1} \left(\sum_{j=s+1}^n \nu_{2^j}^s \hat{f} \right) : n \in \mathbb{N}_0 \right) \right\|_{\ell^p} \leq C_{p,r} s \log(s+2) \|f\|_{\ell^p}.$$

Then interpolation (6.24) with (6.21) will immediately imply (6.20). The proof of (6.24) will consist of two parts. We shall bound separately the variations when $0 \leq n \leq 2^{\kappa_s}$ and when $n \geq 2^{\kappa_s}$, see Theorem 6.2 and Theorem 6.4 respectively.

Theorem 6.2. *Let $p \in (1, \infty)$ and $r \in (2, \infty)$ then there is a constant $C_{p,r} > 0$ such that for every $s \in \mathbb{N}_0$ and every $f \in \ell^p(\mathbb{Z}^d)$ we have*

$$\left\| V_r \left(\mathcal{F}^{-1} \left(\sum_{j=s+1}^n \nu_{2^j}^s \hat{f} \right) : 0 \leq n \leq 2^{\kappa_s} \right) \right\|_{\ell^p} \leq C_{p,r} s \log(s+2) \|f\|_{\ell^p}.$$

Proof. Let $J = \lfloor e^{(s+1)^{1/2}} \rfloor$ and define the multiplier

$$\mu_J(\xi) = J^{-k} \sum_{y \in \mathbb{N}_J^k} e^{2\pi i \xi \cdot \mathcal{Q}(y)}$$

where $\mathbb{N}_J^k = \{1, 2, \dots, J\}^k$. We see that μ_J corresponds to the averaging operator, i.e. $M_J f = \mathcal{F}^{-1}(\mu_J \hat{f})$. Thus for each $r \in [1, \infty]$ we have

$$\|\mathcal{F}^{-1}(\mu_J \hat{f})\|_{\ell^r} \leq \|f\|_{\ell^r}.$$

Moreover, if $\xi \in \mathbb{T}^d$ is such that $|\xi_\gamma - a_\gamma/q| \leq 2^{-s(|\gamma|-\chi)}$ for every $\gamma \in \Gamma$ with some $1 \leq q \leq e^{(s+1)^{1/10}}$ and $a \in A_q$, then

$$\mu_J(\xi) = G(a/q) \Phi_J(\xi - a/q) + \mathcal{O}(e^{-\frac{1}{2}(s+1)^{1/2}}).$$

Indeed, by Proposition 4.1 with $L_1 = 2^s$, $L_2 = 2^{s\chi}$, $L_3 = e^{(s+1)^{1/10}}$ and $N = J$ we see that the error term is dominated by

$$\begin{aligned} L_3/J + L_2 L_3 J^{-1} \sum_{\gamma \in \Gamma} (J/L_1)^{|\gamma|} &\lesssim e^{(s+1)^{1/10} - (s+1)^{1/2}} + 2^{s\chi} e^{(s+1)^{1/10} - (s+1)^{1/2}} (e^{(s+1)^{1/2}} \cdot 2^{-s}) \\ &\lesssim e^{-\frac{1}{2}(s+1)^{1/2}}. \end{aligned}$$

Therefore,

$$(6.25) \quad |\mu_J(\xi) - G(a/q)| \lesssim |G(a/q)(\Phi_J(\xi - a/q) - 1)| + e^{-\frac{1}{2}(s+1)^{1/2}} \lesssim e^{-\frac{1}{2}(s+1)^{1/2}}$$

since

$$|\Phi_J(\xi - a/q) - 1| \lesssim |J^A(\xi - a/q)| \lesssim e^{(s+1)^{1/2}} 2^{-s(1-\chi)}.$$

Let us define the multipliers

$$\Pi_{2^j}^s(\xi) = \sum_{a/q \in \mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_s} (\Psi_{2^j}(\xi - a/q) - \Psi_{2^{j-1}}(\xi - a/q)) \eta(2^{s(A-\chi I)}(\xi - a/q))$$

and observe that by (6.25) we have

$$(6.26) \quad \nu_{2^j}^s(\xi) - \mu_J(\xi) \Pi_{2^j}^s(\xi) = \mathcal{O}(e^{-\frac{1}{2}(s+1)^{1/2}}).$$

By (6.26) and Plancherel's theorem we have

$$(6.27) \quad \|\mathcal{F}^{-1}((\nu_{2^j}^s - \mu_J \Pi_{2^j}^s) \hat{f})\|_{\ell^2} \lesssim e^{-\frac{1}{2}(s+1)^{1/2}} \|f\|_{\ell^2}$$

furthermore, for every $p \in (1, \infty)$ we obtain

$$(6.28) \quad \|\mathcal{F}^{-1}((\nu_{2^j}^s - \mu_J \Pi_{2^j}^s) \hat{f})\|_{\ell^p} \lesssim |U_{(s+1)^l}| \|f\|_{\ell^p} \lesssim e^{(s+1)^{1/10}} \|f\|_{\ell^p}.$$

Interpolating now (6.27) with (6.28) one has for some $c_p > 0$ that

$$(6.29) \quad \|\mathcal{F}^{-1}((\nu_{2^j}^s - \mu_J \Pi_{2^j}^s) \hat{f})\|_{\ell^p} \lesssim e^{-c_p(s+1)^{1/2}} \|f\|_{\ell^p}.$$

Thus by (6.29) we obtain

$$\begin{aligned} & \left\| V_r \left(\mathcal{F}^{-1} \left(\sum_{j=s+1}^n (\nu_{2^j}^s - \mu_J \Pi_{2^j}^s) \hat{f} \right) : 0 \leq n \leq 2^{\kappa_s} \right) \right\|_{\ell^p} \\ & \lesssim \sum_{n=0}^{2^{\kappa_s}} \left\| \mathcal{F}^{-1}((\nu_{2^n}^s - \mu_J \Pi_{2^n}^s) \hat{f}) \right\|_{\ell^p} \lesssim 2^{\kappa_s} e^{-c_p(s+1)^{1/2}} \|f\|_{\ell^p} \lesssim \|f\|_{\ell^p} \end{aligned}$$

since $2^{\kappa_s} e^{-c_p(s+1)^{1/2}} \lesssim 1$. The proof of Theorem 6.2 will be completed if we show

$$\left\| V_r \left(\mathcal{F}^{-1} \left(\sum_{j=s+1}^n \Pi_{2^j}^s \hat{f} \right) : 0 \leq n \leq 2^{\kappa_s} \right) \right\|_{\ell^p} \lesssim \kappa_s \log(s+2) \|f\|_{\ell^p}.$$

Appealing to inequality (2.4) we see that

$$\left\| V_r \left(\mathcal{F}^{-1} \left(\sum_{j=s+1}^n \Pi_{2^j}^s \hat{f} \right) : 0 \leq n \leq 2^{\kappa_s} \right) \right\|_{\ell^p} \lesssim \sum_{i=0}^{\kappa_s} \left\| \left(\sum_{j=0}^{2^{\kappa_s-i}-1} \left| \sum_{m \in I_j^i} \mathcal{F}^{-1}(\Pi_{2^m}^s \hat{f}) \right|^2 \right)^{1/2} \right\|_{\ell^p}$$

where $I_j^i = (j2^i, (j+1)2^i]$. For each $i \in \{0, 1, \dots, \kappa_s\}$ we have by Khinchine's inequality that

$$\left\| \left(\sum_{j=0}^{2^{\kappa_s-i}-1} \left| \sum_{m \in I_j^i} \mathcal{F}^{-1}(\Pi_{2^m}^s \hat{f}) \right|^2 \right)^{1/2} \right\|_{\ell^p} \lesssim \left(\int_0^1 \left\| \sum_{j=0}^{2^{\kappa_s-i}-1} \sum_{m \in I_j^i} \varepsilon_j(\omega) \mathcal{F}^{-1}(\Pi_{2^m}^s \hat{f}) \right\|_{\ell^p}^p d\omega \right)^{1/p}.$$

It suffices to show that for every $i \in \{0, 1, \dots, \kappa_s\}$ and $\omega \in [0, 1]$ we have

$$(6.30) \quad \left\| \sum_{j=0}^{2^{\kappa_s-i}-1} \sum_{m \in I_j^i} \varepsilon_j(\omega) \mathcal{F}^{-1}(\Pi_{2^m}^s \hat{f}) \right\|_{\ell^p} \lesssim \log(s+2) \|f\|_{\ell^p}.$$

For any sequence $\varepsilon = (\varepsilon_j(\omega) : 0 \leq j < 2^{\kappa_s-i})$ with $\varepsilon_j(\omega) \in \{-1, 1\}$, we consider the operator

$$\mathcal{T}_\varepsilon f = \sum_{a/q \in \mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_s} \mathcal{F}^{-1}(\Theta(\cdot - a/q) \eta_s(\cdot - a/q) \hat{f})$$

with

$$\Theta = \sum_{j=0}^{2^{\kappa_s-i}-1} \varepsilon_j(\omega) \sum_{m \in I_j^i} (\Psi_{2^m} - \Psi_{2^{m-1}}).$$

We notice that the multiplier Θ corresponds to a continuous singular Radon transform. Thus Θ defines a bounded operator on $L^r(\mathbb{R}^d)$ for any $r \in (1, \infty)$ with the bound independent of the sequence $(\varepsilon_j(\omega) : 0 \leq j \leq 2^{\kappa_s-i})$ (see [19, Section 11]). Hence, by Theorem 3.2

$$\|\mathcal{T}_\varepsilon f\|_{\ell^p} \lesssim \log(s+2) \|f\|_{\ell^p}$$

and consequently we obtain (6.30) and the proof of Theorem 6.2 is completed. \square

For each $N \in \mathbb{N}$ and $s \in \mathbb{N}_0$ we define multipliers

$$\Omega_N^s(\xi) = \sum_{a/q \in \mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_s^l} G(a/q) \Theta_N(\xi - a/q) \varrho_s(\xi - a/q),$$

where $\varrho_s(\xi) = \eta(Q_{s+1}^{3dA}\xi)$ and $(\Theta_N : N \in \mathbb{N})$ is a sequence of multipliers on \mathbb{R}^d such that for $p \in (1, \infty)$ and $r \in (2, \infty)$ there is a constant $\mathbf{B}_{p,r} > 0$ such that for every $f \in L^2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ we have

$$(6.31) \quad \|V_r(\mathcal{F}^{-1}(\Theta_N \mathcal{F}f) : N \in \mathbb{N})\|_{L^p} \leq \mathbf{B}_{p,r} \|f\|_{L^p}.$$

In fact the multipliers obeying (6.31) have been discussed in the Appendix, see Theorem A.2 in the context of truncated singular integrals. Moreover, in this case $\mathbf{B}_{p,r} \leq \mathbf{B}_{p, \frac{r}{r-2}}$ for some $\mathbf{B}_p > 0$.

Theorem 6.3. *Let $p \in (1, \infty)$ and $r \in (2, \infty)$ then there exists $C_p > 0$ such that for any $s \in \mathbb{N}_0$ and $f \in \ell^p(\mathbb{Z}^d)$ we have*

$$\|V_r(\mathcal{F}^{-1}(\Omega_N^s \hat{f}) : N \in \mathbb{N})\|_{\ell^p} \leq C_p \mathbf{B}_{p,r} \log(s+2) \|f\|_{\ell^p}.$$

Proof. Let us observe that

$$\mathcal{F}^{-1}(\Theta_N(\cdot - a/q) \varrho_s(\cdot - a/q) \hat{f})(Q_s x + m) = \mathcal{F}^{-1}(\Theta_N \varrho_s \hat{f}(\cdot + a/q))(Q_s x + m) e^{-2\pi i(a/q) \cdot m}.$$

Therefore,

$$\|V_r(\mathcal{F}^{-1}(\Omega_N^s \hat{f}) : N \in \mathbb{N})\|_{\ell^p}^p = \sum_{m \in \mathbb{N}_{Q_s}^d} \|V_r(\mathcal{F}^{-1}(\Theta_N \varrho_s F(\cdot; m))(Q_s x + m) : N \in \mathbb{N})\|_{\ell^p(x)}^p$$

where

$$(6.32) \quad F(\xi; m) = \sum_{a/q \in \mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_s^l} G(a/q) \hat{f}(\xi + a/q) e^{-2\pi i(a/q) \cdot m}.$$

Now, by Proposition 3.1 and (6.32) we get

$$\begin{aligned} \sum_{m \in \mathbb{N}_{Q_s}^d} \|V_r(\mathcal{F}^{-1}(\Theta_N \varrho_s F(\cdot; m))(Q_s x + m) : N \in \mathbb{N})\|_{\ell^p(x)}^p \\ \leq C_p^p \mathbf{B}_{p,r}^p \sum_{m \in \mathbb{N}_{Q_s}^d} \|\mathcal{F}^{-1}(\varrho_s F(\cdot; m))(Q_s x + m)\|_{\ell^p(x)}^p \\ = C_p^p \mathbf{B}_{p,r}^p \left\| \sum_{a/q \in \mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_s^l} G(a/q) \mathcal{F}^{-1}(\varrho_s(\cdot - a/q) \hat{f}) \right\|_{\ell^p}^p. \end{aligned}$$

It suffices to prove

$$(6.33) \quad \|\mathcal{F}^{-1}(\tilde{\Pi}_s^G \hat{f})\|_{\ell^p} \lesssim \log(s+2) \|f\|_{\ell^p}$$

where

$$\tilde{\Pi}_s^G(\xi) = \sum_{a/q \in \mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_s^l} G(a/q) \varrho_s(\xi - a/q).$$

Observe that arguing in a similar way as in the proof of Theorem 6.2 we obtain by (6.25) that

$$(6.34) \quad |\tilde{\Pi}_s^G(\xi) - \mu_J(\xi) \tilde{\Pi}_s(\xi)| \lesssim e^{-\frac{1}{2}(s+1)^{1/2}}$$

where

$$\tilde{\Pi}_s(\xi) = \sum_{a/q \in \mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_s^l} \varrho_s(\xi - a/q).$$

Therefore, (6.34) combined with Plancherel's theorem yields

$$(6.35) \quad \|\mathcal{F}^{-1}((\tilde{\Pi}_s^G - \mu_J \tilde{\Pi}_s) \hat{f})\|_{\ell^2} \lesssim e^{-\frac{1}{2}(s+1)^{1/2}} \|f\|_{\ell^2}.$$

We can conclude by interpolation with (6.35) that

$$\|\mathcal{F}^{-1}((\tilde{\Pi}_s^G - \mu_J \tilde{\Pi}_s) \hat{f})\|_{\ell^p} \lesssim \|f\|_{\ell^p}.$$

since by Theorem 3.2 we have

$$\|\mathcal{F}^{-1}(\tilde{\Pi}_s \hat{f})\|_{\ell^p} \lesssim \log(s+2) \|f\|_{\ell^p}$$

and the trivial bound

$$\|\mathcal{F}^{-1}(\tilde{\Pi}_s^G \hat{f})\|_{\ell^p} \lesssim |\mathcal{U}_{(s+1)^l}| \|f\|_{\ell^p} \lesssim e^{(d+1)(s+1)^{1/10}} \|f\|_{\ell^p}.$$

This establishes the bound in (6.33) and the proof of Theorem 6.3 is finished. \square

Theorem 6.4. *Let $p \in (1, \infty)$ and $r \in (2, \infty)$ then there is a constant $C_{p,r} > 0$ such that for every $s \in \mathbb{N}_0$ and $f \in \ell^p(\mathbb{Z}^d)$ we have*

$$\left\| V_r \left(\mathcal{F}^{-1} \left(\sum_{j=s+1}^n \nu_{2^j}^s \hat{f} \right) : n \geq 2^{\kappa_s} \right) \right\|_{\ell^p} \leq C_{p,r} \log(s+2) \|f\|_{\ell^p}.$$

Proof. The proof of Theorem A.6 ensures that the sequence $(\Psi_{2^n} : n \in \mathbb{N}_0)$ satisfies (6.31). Thus in view of Theorem 6.3 which will be applied with $N = 2^n$ and $\Theta_{2^n} = \Psi_{2^n}$ it only suffices to prove that for any $n \geq 2^{\kappa_s}$ we have

$$(6.36) \quad \left\| \mathcal{F}^{-1}((\nu_{2^n}^s - \tilde{\Omega}_{2^n}^s) \hat{f}) \right\|_{\ell^p} \lesssim 2^{-c_p^1 n} e^{c_p^2 (s+1)^{1/10}} \|f\|_{\ell^p}$$

for some $c_p^1, c_p^2 > 0$, where

$$\tilde{\Omega}_{2^j}^s(\xi) = \sum_{a/q \in \mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_{s^l}} G(a/q) (\Psi_{2^j}(\xi - a/q) - \Psi_{2^{j-1}}(\xi - a/q)) \varrho_s(\xi - a/q)$$

and $\Omega_{2^n} = \sum_{j=1}^n \tilde{\Omega}_{2^j}^s$. Obviously we have

$$(6.37) \quad \left\| \mathcal{F}^{-1}((\nu_{2^n}^s - \tilde{\Omega}_{2^n}^s) \hat{f}) \right\|_{\ell^p} \lesssim |\mathcal{U}_{(s+1)^l}| \|f\|_{\ell^p} \lesssim e^{(d+1)(s+1)^{1/10}} \|f\|_{\ell^p}.$$

Next, we observe that $\varrho_s(\xi - a/q) - \eta_s(\xi - a/q) \neq 0$ implies that $|\xi_\gamma - a_\gamma/q| \geq (16d)^{-1} Q_{s+1}^{-3d|\gamma|}$ for some $\gamma \in \Gamma$. Therefore, for $n \geq 2^{\kappa_s}$ we have

$$2^{n|\gamma|} \cdot |\xi_\gamma - a_\gamma/q| \gtrsim 2^{n|\gamma|} Q_{s+1}^{-3d|\gamma|} \gtrsim 2^{n/2},$$

since

$$2^{n/2} Q_{s+1}^{-3d} \geq 2^{2^{\kappa_s}-1} e^{-3d(s+1)^{1/10}} e^{(s+1)^{1/10}} \geq e^{(s+1)^{1/10}}$$

for sufficiently large $s \in \mathbb{N}_0$. Using (6.1), we obtain

$$|\Psi_{2^n}(\xi - a/q) - \Psi_{2^{n-1}}(\xi - a/q)| \lesssim 2^{-n/(2d)}.$$

Hence, by (3.1)

$$\left| \sum_{a/q \in \mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_{s^l}} G(a/q) (\Psi_{2^n}(\xi - a/q) - \Psi_{2^{n-1}}(\xi - a/q)) (\eta_s(\xi - a/q) - \varrho_s(\xi - a/q)) \right| \leq C(s+1)^{-\delta l} 2^{-n/(2d)}.$$

Thus, by Plancherel's theorem we obtain

$$(6.38) \quad \left\| \mathcal{F}^{-1}((\nu_{2^n}^s - \Omega_{2^n}^s) \hat{f}) \right\|_{\ell^2} \lesssim 2^{-n/(2d)} (s+1)^{-\delta l} \|f\|_{\ell^2}.$$

Interpolating now (6.38) with (6.37) we obtain (6.36) and this completes the proof of Theorem 6.4. \square

7. SHORT VARIATION ESTIMATES FOR TRUNCATED SINGULAR INTEGRAL OPERATORS

According to (2.5) and the estimates for long variations from the previous section it remains to prove that for all $p \in (1, \infty)$ there is $C_p > 0$ such that for all $f \in \ell^p(\mathbb{Z}^d)$ with finite support we have

$$\left\| \left(\sum_{n \geq 0} V_2(T_N f : N \in [2^n, 2^{n+1}))^2 \right)^{1/2} \right\|_{\ell^p} \leq C_p \|f\|_{\ell^p}.$$

Using multipliers Ξ_n from (6.3) observe that

$$(7.1) \quad \left\| \left(\sum_{n \geq 0} V_2(T_N f : N \in [2^n, 2^{n+1}))^2 \right)^{1/2} \right\|_{\ell^p} \\ \leq \left\| \left(\sum_{n \geq 0} V_2((T_N - T_{2^n}) \mathcal{F}^{-1}(\Xi_n \hat{f}) : N \in [2^n, 2^{n+1}))^2 \right)^{1/2} \right\|_{\ell^p} \\ + \left\| \left(\sum_{n \geq 0} V_2((T_N - T_{2^n}) \mathcal{F}^{-1}((1 - \Xi_n) \hat{f}) : N \in [2^n, 2^{n+1}))^2 \right)^{1/2} \right\|_{\ell^p}.$$

7.1. The estimate of the second norm in (7.1). We may assume without loss of generality, that $1 < r \leq \min\{2, p\}$, since r -variations are decreasing, and it suffices to show that

$$(7.2) \quad \|V_r((T_N - T_{2^n})\mathcal{F}^{-1}((1 - \Xi_n)\hat{f}) : N \in [2^n, 2^{n+1}))\|_{\ell^p} \lesssim (n+1)^{-2} \|f\|_{\ell^p}.$$

For this purpose we shall use (2.8). Namely, by (2.8) we immediately see that

$$(7.3) \quad \|V_r((T_N - T_{2^n})\mathcal{F}^{-1}((1 - \Xi_n)\hat{f}) : N \in [2^n, 2^{n+1}))\|_{\ell^p} \lesssim \max\{\mathbf{U}_p, 2^{n/r} \mathbf{U}_p^{1-1/r} \mathbf{V}_p^{1/r}\}$$

where

$$\mathbf{U}_p = \sup_{2^n \leq N \leq 2^{n+1}} \|(T_N - T_{2^n})\mathcal{F}^{-1}((1 - \Xi_n)\hat{f})\|_{\ell^p}$$

and

$$\mathbf{V}_p = \sup_{2^n \leq N < 2^{n+1}} \|(T_{N+1} - T_N)\mathcal{F}^{-1}((1 - \Xi_n)\hat{f})\|_{\ell^p}.$$

In view of (6.4) we see that

$$(7.4) \quad \mathbf{U}_p \lesssim \log(n+2) \|f\|_{\ell^p} \quad \text{and} \quad \mathbf{V}_p \lesssim 2^{-n} \log(n+2) \|f\|_{\ell^p}.$$

In fact arguing as in Section 5 we can prove, using Theorem, 3.1 that for big enough $\alpha > 0$, which will be specified later, and for all $n \in \mathbb{N}_0$ and $N \simeq 2^n$ we have

$$(7.5) \quad \mathbf{U}_2 \lesssim (1+n)^{-\alpha} \log(n+2) \|f\|_{\ell^2}.$$

Interpolating (7.5) with (7.4) we obtain

$$(7.6) \quad \mathbf{U}_p \lesssim (1+n)^{-c_p \alpha} \log(n+2) \|f\|_{\ell^p}.$$

for some $c_p > 0$. Choosing $\alpha > 0$ and $l \in \mathbb{N}$ appropriately large we see that (7.6) combined with (7.3) easily imply (7.2).

7.2. The estimate of the first norm in (7.1). By Proposition 6.1 if $\xi \in \mathbb{T}^d$ is such that

$$\left| \xi_\gamma - \frac{a_\gamma}{q} \right| \leq \frac{1}{8d \cdot 2^{n(|\gamma| - \chi)}}$$

for every $\gamma \in \Gamma$ with $1 \leq q \leq e^{n^{1/10}}$ then we have

$$(7.7) \quad m_N(\xi) - m_{2^n}(\xi) = G(a/q) (\Psi_N(\xi - a/q) - \Psi_{2^n}(\xi - a/q)) + q^{-\delta} E_{2^n}(\xi)$$

for every $N \simeq 2^n$, where

$$(7.8) \quad |E_{2^n}(\xi)| \lesssim 2^{-n/2}.$$

Let us introduce for every $j, n \in \mathbb{N}_0$ the new multipliers

$$\Xi_n^j(\xi) = \sum_{a/q \in \mathcal{U}_{n^l}} \eta(2^{nA+jI}(\xi - a/q))$$

and note that

$$\begin{aligned} & \left\| \left(\sum_{n \geq 0} V_2((T_N - T_{2^n})\mathcal{F}^{-1}(\Xi_n \hat{f}) : N \in [2^n, 2^{n+1}))^2 \right)^{1/2} \right\|_{\ell^p} \\ & \leq \left\| \left(\sum_{n \geq 0} V_2((T_N - T_{2^n})\mathcal{F}^{-1} \left(\sum_{-\chi n \leq j < n} (\Xi_n^j - \Xi_n^{j+1}) \hat{f} \right) : N \in [2^n, 2^{n+1}))^2 \right)^{1/2} \right\|_{\ell^p} \\ & \quad + \left\| \left(\sum_{n \geq 0} V_2((T_N - T_{2^n})\mathcal{F}^{-1}(\Xi_n \hat{f}) : N \in [2^n, 2^{n+1}))^2 \right)^{1/2} \right\|_{\ell^p} = I_p^1 + I_p^2. \end{aligned}$$

We will estimate I_p^1 and I_p^2 separately. First, observe that by (7.7) and (7.8), for any $N \simeq 2^n$ and any $a/q \in \mathcal{U}_{n^l}$ we have

$$(7.9) \quad \begin{aligned} |m_N(\xi) - m_{2^n}(\xi)| & \lesssim q^{-\delta} |\Psi_N(\xi - a/q) - \Psi_{2^n}(\xi - a/q)| + q^{-\delta} 2^{-n/2} \\ & \lesssim q^{-\delta} (\min\{1, |2^{nA}(\xi - a/q)|_\infty, |2^{nA}(\xi - a/q)|_\infty^{-1/d}\} + 2^{-n/2}) \end{aligned}$$

where the last bound follows from (6.1) and (6.2). Consequently (7.9) implies

$$(7.10) \quad |(m_N(\xi) - m_{2^n}(\xi))(\eta(2^{nA+jI}(\xi - a/q)) - \eta(2^{nA+(j+1)I}(\xi - a/q)))| \lesssim q^{-\delta} (2^{-|j|/d} + 2^{-n/2}).$$

We begin with bounding I_p^2 . Since r -variations are decreasing we can assume that $1 < r \leq \min\{2, p\}$ and it will suffice to show, for some $\varepsilon = \varepsilon_{p,r} > 0$, that

$$(7.11) \quad \|V_r((T_N - T_{2^n})\mathcal{F}^{-1}(\Xi_n^{\hat{f}}) : N \in [2^n, 2^{n+1}])\|_{\ell^p} \lesssim 2^{-\varepsilon n} \|f\|_{\ell^p}.$$

Exploiting (2.8) we have

$$\|V_r((T_N - T_{2^n})\mathcal{F}^{-1}(\Xi_n^{\hat{f}}) : N \in [2^n, 2^{n+1}])\|_{\ell^p} \lesssim \max\{\mathbf{U}_p, 2^{n/r} \mathbf{U}_p^{1-1/r} \mathbf{V}_p^{1/r}\}$$

where

$$\mathbf{U}_p = \sup_{2^n \leq N \leq 2^{n+1}} \|(T_N - T_{2^n})\mathcal{F}^{-1}(\Xi_n^{\hat{f}})\|_{\ell^p}$$

and

$$\mathbf{V}_p = \sup_{2^n \leq N < 2^{n+1}} \|(T_{N+1} - T_N)\mathcal{F}^{-1}(\Xi_n^{\hat{f}})\|_{\ell^p}.$$

In view of Theorem 3.2 we see that

$$(7.12) \quad \mathbf{U}_p \lesssim \log(n+2) \|f\|_{\ell^p} \quad \text{and} \quad \mathbf{V}_p \lesssim 2^{-n} \log(n+2) \|f\|_{\ell^p}.$$

For $p = 2$ by Plancherel's theorem and (7.9) we obtain

$$(7.13) \quad \begin{aligned} & \|(T_N - T_{2^n})\mathcal{F}^{-1}(\Xi_n^{\hat{f}})\|_{\ell^2} \\ &= \left(\int_{\mathbb{T}^d} \sum_{a/q \in \mathcal{Q}_{n,l}} |m_N(\xi) - m_{2^n}(\xi)|^2 \eta(2^{nA+nI}(\xi - a/q))^2 |\hat{f}(\xi)|^2 d\xi \right)^{1/2} \lesssim 2^{-n/(2d)} \|f\|_{\ell^2}. \end{aligned}$$

Therefore, interpolating (7.12) with (7.13) we obtain for every $p \in (1, \infty)$ that

$$\mathbf{U}_p \lesssim 2^{-\varepsilon n} \|f\|_{\ell^p}$$

which in turn implies (7.11) and $I_p^2 \lesssim \|f\|_{\ell^p}$.

We shall now estimate I_p^1 , for this purpose, for any $0 \leq s < n$, we define

$$\Delta_{n,s}^j(\xi) = \sum_{a/q \in \mathcal{Q}_{(s+1)l} \setminus \mathcal{Q}_{s,l}} (\eta(2^{nA+jI}(\xi - a/q)) - \eta(2^{nA+(j+1)I}(\xi - a/q))) \eta(2^{s(A-\chi I)}(\xi - a/q))$$

hence

$$\Xi_n^j(\xi) - \Xi_n^{j+1}(\xi) = \sum_{0 \leq s < n} \Delta_{n,s}^j(\xi).$$

Now we see that

$$\begin{aligned} I_p^1 &= \left\| \left(\sum_{n \geq 0} V_2((T_N - T_{2^n})\mathcal{F}^{-1} \left(\sum_{-\chi n \leq j < n} \sum_{0 \leq s < n} \Delta_{n,s}^j \hat{f} \right) : N \in [2^n, 2^{n+1}]) \right)^2 \right\|_{\ell^p}^{1/2} \\ &\leq \sum_{s \geq 0} \sum_{j \in \mathbb{Z}} \left\| \left(\sum_{n \geq \max\{s, j, -j/\chi\}} V_2((T_N - T_{2^n})\mathcal{F}^{-1}(\Delta_{n,s}^j \hat{f}) : N \in [2^n, 2^{n+1}]) \right)^2 \right\|_{\ell^p}^{1/2}. \end{aligned}$$

The task now is to show that for some $\varepsilon_p > 0$

$$(7.14) \quad J_p = \left\| \left(\sum_{n \geq \max\{s, j, -j/\chi\}} V_2((T_N - T_{2^n})\mathcal{F}^{-1}(\Delta_{n,s}^j \hat{f}) : N \in [2^n, 2^{n+1}]) \right)^2 \right\|_{\ell^p}^{1/2} \lesssim s^{-2} 2^{-\varepsilon_p |j|} \|f\|_{\ell^p}.$$

To estimate (7.14) we will use (2.6) from Lemma 2.2 with $r = 2$. Namely, let $h_{j,s} = 2^{\varepsilon |j|} (s+1)^\tau$ with some $\varepsilon > 0$ and $\tau > 2$ which we choose later. Then for some $2^n \leq t_0 < t_1 < \dots < t_h < 2^{n+1}$ such that $t_{v+1} - t_v \simeq 2^n/h$ where $h = \min\{h_{j,s}, 2^n\}$ we have

$$(7.15) \quad \begin{aligned} J_p &\lesssim \left\| \left(\sum_{n \geq \max\{s, j, -j/\chi\}} \sum_{v=0}^h |(T_{2^{t_v}} - T_{2^n})\mathcal{F}^{-1}(\Delta_{n,s}^j \hat{f})|^2 \right)^{1/2} \right\|_{\ell^p} \\ &\quad + \left\| \left(\sum_{n \geq \max\{s, j, -j/\chi\}} \sum_{v=0}^{h-1} \left(\sum_{u=t_v}^{t_{v+1}-1} |(T_{u+1} - T_u)\mathcal{F}^{-1}(\Delta_{n,s}^j \hat{f})|^2 \right)^{1/2} \right) \right\|_{\ell^p} = J_p^1 + J_p^2. \end{aligned}$$

7.2.1. *The estimates for J_p^1 .* We begin with $p = 2$ and show that

$$(7.16) \quad J_2^1 \lesssim 2^{-|j|(1/(2d)-\varepsilon/2)}(s+1)^{-\delta l+\tau/2} \|f\|_{\ell^2}.$$

For the simplicity of notation define

$$\varrho_{n,j}(\xi) = (\eta(2^{nA+jI}\xi) - \eta(2^{nA+(j+1)I}\xi))\eta(2^{s(A-\chi I)}\xi).$$

By Plancherel's theorem and (7.10) we have

$$\begin{aligned} J_2^1 &= \left(\sum_{v=0}^h \int_{\mathbb{T}^d} \sum_{a/q \in \mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_{s^l}} \sum_{n \geq \max\{s,j,-j/\chi\}} |m_{2^{t_v}}(\xi) - m_{2^n}(\xi)|^2 \varrho_{n,j}(\xi - a/q)^2 |\hat{f}(\xi)|^2 d\xi \right)^{1/2} \\ &\lesssim \left(\sum_{v=0}^h 2^{-|j|/d} \int_{\mathbb{T}^d} \sum_{a/q \in \mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_{s^l}} q^{-2\delta} \eta(2^{s(A-\chi I)}(\xi - a/q)) |\hat{f}(\xi)|^2 d\xi \right)^{1/2} \\ &\lesssim h^{1/2} 2^{-|j|/(2d)} \left(\int_{\mathbb{T}^d} (s+1)^{-2\delta l} \sum_{a/q \in \mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_{s^l}} \eta(2^{s(A-\chi I)}(\xi - a/q)) |\hat{f}(\xi)|^2 d\xi \right)^{1/2} \\ &\lesssim h^{1/2} 2^{-|j|/(2d)} (s+1)^{-\delta l} \|f\|_{\ell^2} \end{aligned}$$

as desired. We have used the fact that $q \gtrsim (s+1)^l$ whenever $a/q \in \mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_{s^l}$ and

$$\sum_{n \geq \max\{s,j,-j/\chi\}} \varrho_{n,j}(\xi) \lesssim \eta(2^{s(A-\chi I)}(\xi - a/q))$$

and the disjointness of $\eta(2^{s(A-\chi I)}(\cdot - a/q))$ while a/q varies over $\mathcal{U}_{(s+1)^k} \setminus \mathcal{U}_{s^k}$.

Moreover, for any $p \in (1, \infty)$ we have

$$(7.17) \quad J_p^1 \lesssim 2^{\varepsilon|j|/2} (s+1)^{\tau/2} \log(s+2) \|f\|_{\ell^p}.$$

Indeed, appealing to the vector-valued inequality for the maximal function corresponding to the averaging operators from [15] we see that

$$\begin{aligned} J_p^1 &= \left\| \left(\sum_{n \geq \max\{s,j,-j/\chi\}} \sum_{v=0}^h |(T_{2^{t_v}} - T_{2^n})\mathcal{F}^{-1}(\Xi_{n,s}^j \hat{f})|^2 \right)^{1/2} \right\|_{\ell^p} \\ &\lesssim h^{1/2} \left\| \left(\sum_{n \geq \max\{s,j,-j/\chi\}} \sup_{N \in \mathbb{N}} M_N(|\mathcal{F}^{-1}(\Xi_{n,s}^j \hat{f})|)^2 \right)^{1/2} \right\|_{\ell^p} \\ &\lesssim h^{1/2} \left\| \left(\sum_{n \geq \max\{s,j,-j/\chi\}} |\mathcal{F}^{-1}(\Xi_{n,s}^j \hat{f})|^2 \right)^{1/2} \right\|_{\ell^p} \lesssim h^{1/2} \log(s+2) \|f\|_{\ell^p}, \end{aligned}$$

where in the last step we have used (5.17). Interpolating now (7.17) with better (7.16) estimate we obtain for some $\varepsilon_p > 0$ that

$$J_p^1 \lesssim (s+1)^{-2} 2^{-\varepsilon_p|j|} \|f\|_{\ell^p}.$$

The proof of (7.15) will be completed if we obtain the same kind of bound for J_p^2 .

7.2.2. *The estimates for J_p^2 .* We begin with $p = 2$ and our aim will be to show

$$(7.18) \quad J_2^2 \lesssim 2^{-\varepsilon|j|/2} (s+1)^{-\tau/2} \|f\|_{\ell^2}.$$

Since $t_{v+1} - t_v \simeq 2^n/h$ then by the Cauchy-Schwarz inequality we obtain

$$J_2^2 \leq \left\| \left(\sum_{n \geq \max\{s,j,-j/\chi\}} 2^n/h \sum_{u=2^n}^{2^{n+1}-1} |(T_{u+1} - T_u)\mathcal{F}^{-1}(\Xi_{n,s}^j \hat{f})|^2 \right)^{1/2} \right\|_{\ell^2}.$$

By (7.10) we have for $u \simeq 2^n$ that

$$(7.19) \quad |m_{u+1}(\xi) - m_u(\xi)| \lesssim \min \{2^{-n}, q^{-\delta}(2^{-|j|/d} + 2^{-n/2})\}.$$

Two cases must be distinguished. Assume now that $h = 2^{\varepsilon|j|}(s+1)^\tau$, therefore, again by Plancherel's theorem, we obtain

$$\begin{aligned} J_2^2 &\leq \left(\sum_{n \geq \max\{s, j, -j/\chi\}} \int_{\mathbb{T}^d} 2^n/h \sum_{u=2^n}^{2^{n+1}-1} \sum_{a/q \in \mathcal{U}_{(s+1)^k} \setminus \mathcal{U}_{s^k}} |m_{u+1}(\xi) - m_u(\xi)|^2 \varrho_{n,j}(\xi - a/q)^2 |\hat{f}(\xi)|^2 d\xi \right)^{1/2} \\ &\lesssim h^{-1/2} \left(\int_{\mathbb{T}^d} \sum_{a/q \in \mathcal{U}_{(s+1)^k} \setminus \mathcal{U}_{s^k}} \sum_{n \geq \max\{s, j, -j/\chi\}} \varrho_{n,j}(\xi - a/q) |\hat{f}(\xi)|^2 d\xi \right)^{1/2} \lesssim h^{-1/2} \|f\|_{\ell^2} \end{aligned}$$

since by the telescoping nature and the disjointness of supports when a/q varies over $\mathcal{U}_{(s+1)^k} \setminus \mathcal{U}_{s^k}$ we have

$$\sum_{a/q \in \mathcal{U}_{(s+1)^k} \setminus \mathcal{U}_{s^k}} \sum_{n \geq \max\{s, j, -j/\chi\}} \varrho_{n,j}(\xi - a/q) \lesssim \sum_{a/q \in \mathcal{U}_{(s+1)^k} \setminus \mathcal{U}_{s^k}} \eta(2^{s(A-\chi I)}(\xi - a/q)) \lesssim 1.$$

If $h = 2^n$ then by (7.19) we get

$$\begin{aligned} J_2^2 &\leq \left(\sum_{n \geq \max\{s, j, -j/\chi\}} \int_{\mathbb{T}^d} \sum_{u=2^n}^{2^{n+1}-1} \sum_{a/q \in \mathcal{U}_{(s+1)^k} \setminus \mathcal{U}_{s^k}} |m_{u+1}(\xi) - m_u(\xi)|^2 \varrho_{n,j}(\xi - a/q)^2 |\hat{f}(\xi)|^2 d\xi \right)^{1/2} \\ &\lesssim 2^{-|j|/4} 2^{-s/4} \left(\int_{\mathbb{T}^d} \sum_{a/q \in \mathcal{U}_{(s+1)^k} \setminus \mathcal{U}_{s^k}} \sum_{n \geq \max\{s, j, -j/\chi\}} \varrho_{n,j}(\xi - a/q) |\hat{f}(\xi)|^2 d\xi \right)^{1/2} \\ &\lesssim 2^{-|j|/4} 2^{-s/4} \|f\|_{\ell^2} \end{aligned}$$

and (7.18) is proven.

For $p \in (1, \infty)$ we shall prove that

$$(7.20) \quad J_p^2 \lesssim \log(s+2) \|f\|_{\ell^p}.$$

Indeed,

$$\begin{aligned} J_p^2 &= \left\| \left(\sum_{n \geq \max\{s, j, -j/\chi\}} \sum_{v=0}^{h-1} \left(\sum_{u=t_v}^{t_{v+1}-1} |(T_{u+1} - T_u) \mathcal{F}^{-1}(\Xi_{n,s}^j \hat{f})| \right)^2 \right)^{1/2} \right\|_{\ell^p} \\ &\leq \left\| \left(\sum_{n \geq \max\{s, j, -j/\chi\}} \left(\sum_{u=2^n}^{2^{n+1}-1} |(H_{u+1} - H_u) * (\mathcal{F}^{-1}(\Xi_{n,s}^j \hat{f}))|^2 \right)^{1/2} \right) \right\|_{\ell^p} \\ &\lesssim \left\| \left(\sum_{n \geq \max\{s, j, -j/\chi\}} \sup_{N \in \mathbb{N}} M_N(|\mathcal{F}^{-1}(\Xi_{n,s}^j \hat{f})|^2) \right)^{1/2} \right\|_{\ell^p} \\ &\lesssim \left\| \left(\sum_{n \geq \max\{s, j, -j/\chi\}} |\mathcal{F}^{-1}(\Xi_{n,s}^j \hat{f})|^2 \right)^{1/2} \right\|_{\ell^p} \lesssim \log(s+2) \|f\|_{\ell^p}, \end{aligned}$$

where in the penultimate line we have used vector-valued maximal estimates corresponding to the averaging operators from [15] and in the last line we invoked (5.17). Interpolating now the estimate (7.20) with the estimate from (7.18) we obtain for some $\varepsilon_p > 0$ that

$$J_p^2 \lesssim (s+1)^{-2} 2^{\varepsilon_p j} \|f\|_{\ell^p}$$

and the proof of (7.15) is completed.

8. PROOF OF THEOREM D

The definition of r -variations for any $r \in [1, \infty)$ can be extended to more general sets. Namely, let $A \subseteq \mathbb{R}$ and let us define for each $(a_t : t \in A) \subseteq \mathbb{C}$, the r -variational seminorm by setting

$$V_r(a_t : t \in A) = \sup_{\substack{t_0 < t_1 < \dots < t_J \\ t_j \in A}} \left(\sum_{j=1}^J |a_{t_j} - a_{t_{j-1}}|^r \right)^{1/r}$$

where the supremum is taken over all finite increasing sequences $t_0 < t_1 < \dots < t_J$ and $t_j \in A$ for $0 \leq j \leq J$.

Lemma 8.1. *Assume that $r \in [1, \infty)$ and $(a_t : t \in A) \subseteq \mathbb{C}$. Given an increasing sequence of real numbers $(w_k : k \in \mathbb{N})$ we have*

$$(8.1) \quad V_r(a_t : t > 0) \lesssim_r V_r(a_{w_k} : k \in \mathbb{N}) + \left(\sum_{k \in \mathbb{N}} V_r(a_t : t \in [w_k, w_{k+1}))^2 \right)^{1/2}.$$

Proof. For any increasing sequence $t_0 < t_1 < \dots < t_J$ we define $W_j = \{k \in \mathbb{N} : t_j < w_k \leq t_{j+1}\}$ for $0 \leq j \leq J$. If $W_j \neq \emptyset$ then we take $u_j = \min W_j$ and $v_j = \max W_j$. Now if $W_j = \emptyset$ then the term

$$|a_{t_{j+1}} - a_{t_j}|^r$$

is part of the second term in (8.1). If $W_j \neq \emptyset$ then

$$|a_{t_{j+1}} - a_{t_j}|^r \leq 3^r (|a_{t_{j+1}} - a_{v_j}|^r + |a_{v_j} - a_{u_j}|^r + |a_{t_j} - a_{u_j}|^r)$$

and we see that the first and the third terms are part of the square function in (8.1), whereas the middle term is part of the r -variations along $(w_k : k \in \mathbb{N})$. \square

Proof of Theorem D. Define the set $\mathbb{L} = \{x > 0 : x^2 \in \mathbb{N}\}$. The methods presented in the previous sections allow us to establish that for every $p \in (1, \infty)$ and $r \in (2, \infty)$ there is $C_{p,r} > 0$ such that for all $f \in \ell^p(\mathbb{Z}^{d_0})$

$$(8.2) \quad \|V_r(M_N^{\mathcal{P}} f : N \in \mathbb{L})\|_{\ell^p} + \|V_r(T_N^{\mathcal{P}} f : N \in \mathbb{L})\|_{\ell^p} \leq C_{p,r} \|f\|_{\ell^p}.$$

As before, the constant $C_{p,r} \leq C_p \frac{r}{r-2}$ for some $C_p > 0$ which is independent of the coefficients of the polynomial mapping \mathcal{P} .

Then in view of Lemma 8.1 with $w_n = n^{1/2}$ we have

$$(8.3) \quad \|V_r(R_t^{\mathcal{P}} f : t > 0)\|_{\ell^p} \lesssim_r \|V_r(R_N^{\mathcal{P}} f : N \in \mathbb{L})\|_{\ell^p} + \left(\sum_{n \in \mathbb{N}} \|V_r(R_t^{\mathcal{P}} f : t \in [n^{1/2}, (n+1)^{1/2}))\|_{\ell^p}^2 \right)^{1/2}$$

where $R_t^{\mathcal{P}}$ is either $M_t^{\mathcal{P}}$ or $T_t^{\mathcal{P}}$. According to (8.2) we have

$$\|V_r(R_N^{\mathcal{P}} f : N \in \mathbb{L})\|_{\ell^p} \lesssim_{p,r} \|f\|_{\ell^p}.$$

In order to estimate the square function in (8.3), note that the function $t \rightarrow R_t^{\mathcal{P}}$ is constant when $t \in [n^{1/2}, (n+1)^{1/2})$ thus

$$\|V_r(R_t^{\mathcal{P}} f : t \in [n^{1/2}, (n+1)^{1/2}))\|_{\ell^p} \leq \|R_{(n+1)^{1/2}}^{\mathcal{P}} f - R_{n^{1/2}}^{\mathcal{P}} f\|_{\ell^p} \lesssim n^{-1} \|M_{(n+1)^{1/2}} f\|_{\ell^p} \lesssim n^{-1} \|f\|_{\ell^p}.$$

Therefore,

$$\left(\sum_{n \in \mathbb{N}} \|V_r(R_t^{\mathcal{P}} f : t \in [n^{1/2}, (n+1)^{1/2}))\|_{\ell^p}^2 \right)^{1/2} \lesssim \left(\sum_{n \in \mathbb{N}} n^{-2} \right)^{1/2} \|f\|_{\ell^p} \lesssim \|f\|_{\ell^p}.$$

This completes the proof of the theorem. \square

APPENDIX A. VARIATIONAL ESTIMATES FOR THE CONTINUOUS ANALOGUES

This section is intended to provide r -variational estimates for averaging and truncated singular operators of Radon type in the continuous settings. These kinds of questions were extensively discussed in [10], see also the references given there. Here we propose a different approach. Firstly, we will discuss long variations estimates. We give a new proof of Lépingle's inequality which will be very much in spirit of good- λ inequalities. Secondly, we present a new approach to short variation estimates which is based on vector-valued bounds in [15]. This observation, as far as we know, has not been used in this context before. To fix notation let $\mathcal{P} = (\mathcal{P}_1, \dots, \mathcal{P}_{d_0}) : \mathbb{R}^k \rightarrow \mathbb{R}^{d_0}$ be a polynomial mapping whose components \mathcal{P}_j are real valued polynomials on \mathbb{R}^k such that $\mathcal{P}_j(0) = 0$. One of the main objects of our interest will be

$$\mathcal{M}_t^{\mathcal{P}} f(x) = \frac{1}{|G_t|} \int_{G_t} f(x - \mathcal{P}(y)) \, dy$$

for $x \in \mathbb{R}^{d_0}$ where G is an open bounded convex subset of \mathbb{R}^k , containing the origin and

$$G_t = \{x \in \mathbb{R}^k : t^{-1}x \in G\}.$$

For any $r \in [1, \infty)$ the r -variational seminorm V_r of complex-valued functions $(a_t(x) : t > 0)$ is defined by

$$V_r(a_t(x) : t > 0) = \sup_{0 < t_0 < \dots < t_J} \left(\sum_{j=0}^J |a_{t_{j+1}}(x) - a_{t_j}(x)|^r \right)^{1/r}$$

where the supremum is taken over all finite increasing sequences. In order to avoid some problems with measurability of $V_r(a_t(x) : t > 0)$ we assume that $(0, \infty) \ni t \mapsto a_t(x)$ is always a continuous function for every $x \in \mathbb{R}^{d_0}$. The main result of this section is the following theorem.

Theorem A.1. *For every $1 < p < \infty$ and $r \in (2, \infty)$ there is $C_{p,r} > 0$ such that for all $f \in L^p(\mathbb{R}^{d_0})$*

$$\|V_r(\mathcal{M}_t^{\mathcal{P}} f : t > 0)\|_{L^p} \leq C_{p,r} \|f\|_{L^p}.$$

Moreover, the constant $C_{p,r} \leq C_p \frac{r}{r-2}$ for some $C_p > 0$ which is independent of the coefficients of the polynomial mapping \mathcal{P} .

Suppose that $K \in \mathcal{C}^1(\mathbb{R}^k \setminus \{0\})$ is a Calderón–Zygmund kernel satisfying the differential inequality

$$(A.1) \quad |y|^k |K(y)| + |y|^{k+1} |\nabla K(y)| \leq 1$$

for all $y \in \mathbb{R}^k \setminus \{0\}$ and the cancellation condition

$$\int_{G_t \setminus G_s} K(y) \, dy = 0$$

for every $t > s > 0$. We consider a truncated singular Radon transform defined by

$$\mathcal{T}_t^{\mathcal{P}} f(x) = \int_{G_t^c} f(x - \mathcal{P}(y)) K(y) \, dy$$

for $x \in \mathbb{R}^{d_0}$ and $t > 0$. The second main result is the following theorem.

Theorem A.2. *For every $p \in (1, \infty)$ and $r \in (2, \infty)$ there is $C_{p,r} > 0$ such that for all $f \in L^p(\mathbb{R}^{d_0})$*

$$(A.2) \quad \|\mathcal{T}_t^{\mathcal{P}} f : t > 0\|_{L^p} \leq C_{p,r} \|f\|_{L^p}.$$

Moreover, the constant $C_{p,r} \leq C_p \frac{r}{r-2}$ for some $C_p > 0$ which is independent of the coefficients of the polynomial mapping \mathcal{P} .

We immediately see that (A.2) remains true for the operator

$$\tilde{\mathcal{T}}_t^{\mathcal{P}} f(x) = \text{p.v.} \int_{G_t} f(x - \mathcal{P}(y)) K(y) \, dy.$$

We set

$$N_0 = \max \{ \deg \mathcal{P}_j : 1 \leq j \leq d_0 \}.$$

It is convenient to work with the set

$$\Gamma = \{ \gamma \in \mathbb{Z}^k \setminus \{0\} : 0 \leq \gamma_j \leq N_0 \text{ for each } j = 1, \dots, k \}$$

with the lexicographic order. Then each \mathcal{P}_j can be expressed as

$$\mathcal{P}_j(x) = \sum_{\gamma \in \Gamma} c_j^\gamma x^\gamma$$

for some $c_j^\gamma \in \mathbb{R}$. Let us denote by d the cardinality of the set Γ . We identify \mathbb{R}^d with the space of all vectors whose coordinates are labeled by multi-indices $\gamma \in \Gamma$. Let A be a diagonal $d \times d$ matrix such that

$$(Av)_\gamma = |\gamma| v_\gamma.$$

For $t > 0$ we set

$$t^A = \exp(A \log t)$$

i.e. $t^A x = (t^{|\gamma|} x_\gamma : \gamma \in \Gamma)$ for any $x \in \mathbb{R}^d$. Next, we introduce the *canonical* polynomial mapping

$$\mathcal{Q} = (\mathcal{Q}_\gamma : \gamma \in \Gamma) : \mathbb{Z}^k \rightarrow \mathbb{Z}^d$$

where $\mathcal{Q}_\gamma(x) = x^\gamma$ and $x^\gamma = x_1^{\gamma_1} \cdots x_k^{\gamma_k}$. The coefficients $(c_j^\gamma : \gamma \in \Gamma, j \in \{1, \dots, d_0\})$ define a linear transformation $L : \mathbb{R}^d \rightarrow \mathbb{R}^{d_0}$ such that $L\mathcal{Q} = \mathcal{P}$. Indeed, it is enough to set

$$(Lv)_j = \sum_{\gamma \in \Gamma} c_j^\gamma v_\gamma$$

for each $j \in \{1, \dots, d_0\}$ and $v \in \mathbb{R}^d$. Now, proceeding as in Lemma 2.3 we can reduce the matters (see also [5] or [19, p. 515]) to the canonical polynomial mapping. To simplify the notation we will write $\mathcal{M}_t = \mathcal{M}_t^Q$ and $\mathcal{T}_t = \mathcal{T}_t^Q$.

A.1. Long variations. In this subsection we give a new proof of Lépingle's inequality. Since we will appeal to the results from [10] we are going to follow their notation and therefore we will work with a more general setup than it is necessary for our further purposes.

We consider a slightly more general dilation structure

$$t^A = \exp(A \log t)$$

for any $t > 0$, where A is a $d \times d$ matrix whose eigenvalues have positive real parts. We say that any regular quasi-norm $\rho : \mathbb{R}^d \rightarrow [0, \infty)$ is homogeneous with respect to the dilations $(t^A : t > 0)$ if $\rho(t^A x) = t\rho(x)$ for any $x \in \mathbb{R}^d$ and $t > 0$. Here \mathbb{R}^d endowed with a quasi-norm ρ and the Lebesgue measure will be considered as a space of homogeneous type with the quasi-metric induced by ρ .

In this setting let us recall Christ's construction of dyadic cubes [3].

Lemma A.1 ([3]). *There exists a collection of open sets $\{Q_\alpha^k : k \in \mathbb{Z} \text{ and } \alpha \in I_k\}$ and constants $D > 1$, $\delta, \eta > 0$ and $0 < C_1, C_2$ such that*

- (i) $|\mathbb{R}^d \setminus \bigcup_{\alpha \in I_k} Q_\alpha^k| = 0$ for all $k \in \mathbb{Z}$;
- (ii) if $l \leq k$ then either $Q_\beta^l \subseteq Q_\alpha^k$ or $Q_\beta^l \cap Q_\alpha^k = \emptyset$;
- (iii) for each (l, β) and $l \leq k$, there exists a unique α such that $Q_\beta^l \subseteq Q_\alpha^k$;
- (iv) each Q_α^k contains some ball $B(z_\alpha^k, \delta D^k)$ and $\text{diam}(Q_\alpha^k) \leq C_1 D^k$;
- (v) for each (α, k) and $t > 0$ we have $|\{x \in Q_\alpha^k : \text{dist}(x, \mathbb{R}^d \setminus Q_\alpha^k) \leq t D^k\}| \leq C_2 t^\eta |Q_\alpha^k|$.

Now two comments are in order. Firstly, each cube Q_α^k contains a ball and is contained in some ball, each with radius $\simeq D^k$. Secondly, the quasi-metric is translation invariant thus we see that for each (α, k) the measure of Q_α^k is $\simeq D^{\text{tr}(A)k}$. In particular, there is $R > 0$ such that for all $k \in \mathbb{Z}$, $\alpha \in I_k$ and $\beta \in I_{k+1}$

$$(A.3) \quad |Q_\alpha^{k+1}| \leq R |Q_\beta^k|.$$

The collection $\{Q_\alpha^k : k \in \mathbb{Z} \text{ and } \alpha \in I_k\}$ will be called the collection of dyadic cubes in \mathbb{R}^d adapted to the dilation group $(t^A : t > 0)$. In view of Lemma A.1, it gives rise to an atomic filtration. Namely, for each $k \in \mathbb{Z}$ let $\mathcal{F}_k = \sigma(\{Q_\alpha^l : \alpha \in I_l \text{ and } l \geq -k\})$ be the σ -algebra generated by the cubes at level at least $-k$. Then

$$\mathcal{F}_k \subset \mathcal{F}_{k+1}.$$

For a locally integrable function f we set

$$\mathbb{E}_k f(x) = \mathbb{E}[f | \mathcal{F}_k](x) = \frac{1}{|Q_\alpha^k|} \int_{Q_\alpha^k} f(y) dy$$

provided Q_α^k is the unique dyadic cube containing $x \in \mathbb{R}^d$. Thanks to Lemma A.1 (i), it is true for almost all x . We define a martingale difference by

$$\mathbb{D}_k f = \mathbb{E}_k f - \mathbb{E}_{k-1} f$$

Finally, the maximal function and the square function are given by

$$Mf = \sup_{k \in \mathbb{Z}} |\mathbb{E}_k f| \quad \text{and} \quad Sf = \left(\sum_{k \in \mathbb{Z}} |\mathbb{D}_k f|^2 \right)^{1/2},$$

respectively.

The variational estimate for $(\mathbb{E}_k f : k \in \mathbb{Z})$ follow from estimates on λ -jump function J_λ , see [18, 1, 10]. Recall, that for a sequence of complex numbers $(a_j : j \in \mathbb{Z})$ the function $J_\lambda(a_j : j \in \mathbb{Z})$ is equal to the supremum over all $J \in \mathbb{N}$ for which there exists a sequence of integers $t_1 < t_2 < \dots < t_J$ so that

$$|a_{t_{j+1}} - a_{t_j}| > \lambda$$

for every $j = 1, 2, \dots, J-1$. We immediately see that $J_\lambda(a_j : j \in \mathbb{Z}) \leq \lambda^{-r} V_r(a_j : j \in \mathbb{Z})^r$.

Theorem A.3 ([18, 1]). *For each $p \geq 1$ there exists $B_p > 0$ such that for all $f \in L^p(\mathbb{R}^d)$ and $\lambda > 0$, if $p \in (1, \infty)$*

$$\left\| \lambda (J_\lambda(\mathbb{E}_k f : k \in \mathbb{Z}))^{1/2} \right\|_{L^p} \leq B_p \|f\|_{L^p},$$

and if $p = 1$ then for any $t > 0$ we have

$$|\{x \in \mathbb{R}^d : \lambda(J_\lambda(\mathbb{E}_k f(x) : k \in \mathbb{Z}))^{1/2} > t\}| \leq B_1 t^{-1} \|f\|_{L^1}.$$

The next theorem is a new ingredient in proving Lépingle's inequality and is inspired by [7].

Theorem A.4. *For each $q \geq 2$ there is $C_q > 0$ such that for all $r > 2$ and $\lambda > 0$*

$$(A.4) \quad C_q \cdot |\{x \in \mathbb{R}^d : V_r(\mathbb{E}_k f(x) : k \in \mathbb{Z}) > \lambda \text{ and } Mf(x) < \lambda/2\}| \\ \leq |\{x \in \mathbb{R}^d : Sf(x) > \lambda\}| + \lambda^{-q} (r-2)^{-q/2} \int_{\{Sf \leq \lambda\}} Sf(x)^q dx.$$

Proof. By homogeneity, it suffices to prove the result with $\lambda = 1$. Let $B = \{x \in \mathbb{R}^d : Sf(x) > 1\}$, $B^* = \{x \in \mathbb{R}^d : M\mathbb{1}_B(x) > 1/(2R)\}$ and $G = (B^*)^c$. By the maximal inequality, we have

$$|B^*| = |\{x \in \mathbb{R}^d : M\mathbb{1}_B(x) > 1/(2R)\}| \lesssim \int_{\mathbb{R}^d} \mathbb{1}_B(x)^2 dx = |\{x \in \mathbb{R}^d : Sf(x) > 1\}|.$$

Therefore, it is enough to show that

$$|\{x \in G : V_r(\mathbb{E}_k f(x) : k \in \mathbb{Z}) > 1 \text{ and } Mf(x) < 1/2\}| \lesssim (r-2)^{-q/2} \int_{B^c} Sf(x)^q dx.$$

We can pointwise dominate the variation (see [1])

$$V_r(\mathbb{E}_k f : k \in \mathbb{Z})^r \leq \sum_{l \in \mathbb{Z}} 2^{rl} J_{2^l}(\mathbb{E}_k f : k \in \mathbb{Z}).$$

Since $Mf < 1/2$, the above sum runs over $l \leq 0$, which leads to the containment

$$(A.5) \quad \{x \in G : V_r(\mathbb{E}_k f(x) : k \in \mathbb{Z}) > 1 \text{ and } Mf(x) < 1/2\} \\ \subseteq \left\{x \in G : \sum_{l \leq 0} 2^{rl} J_{2^l}(\mathbb{E}_k f(x) : k \in \mathbb{Z}) > 1\right\}.$$

For each $n \in \mathbb{Z}$ we define $U_n = \{x \in \mathbb{R}^d : \mathbb{E}_n \mathbb{1}_B(x) \leq 1/2\}$. Notice that, if $x \in G$ then $x \in U_n$ for all $n \in \mathbb{Z}$. Let

$$g(x) = \sum_{n \in \mathbb{Z}} \mathbb{D}_n f(x) \cdot \mathbb{1}_{U_{n-1}}(x).$$

We observe that $\mathbb{E}_n g(x) = \mathbb{E}_n f(x)$ for all $x \in G$ and $n \in \mathbb{Z}$. Indeed, $\mathbb{D}_n f \cdot \mathbb{1}_{U_{n-1}}$ is \mathcal{F}_n -measurable and

$$\mathbb{E}_m(\mathbb{D}_n f \cdot \mathbb{1}_{U_{n-1}}) = 0$$

for every $m \leq n-1$. Thus, for $x \in G$ we have

$$\mathbb{E}_m g(x) = \sum_{n \leq m} \mathbb{D}_n f(x) \cdot \mathbb{1}_{U_{n-1}}(x) = \mathbb{E}_m f(x).$$

Hence, by (A.5) and Hölder's inequality with $a = \frac{q}{2}$ and $a' = \frac{q}{q-2}$, we obtain

$$\{x \in G : V_r(\mathbb{E}_k f(x) : k \in \mathbb{Z}) > 1 \text{ and } Mf(x) < 1/2\} \\ \subseteq \left\{x \in \mathbb{R}^d : \sum_{l \leq 0} 2^{rl} J_{2^l}(\mathbb{E}_k g(x) : k \in \mathbb{Z}) > 1\right\} \\ \subseteq \left\{x \in G : \left(\sum_{l \leq 0} 2^{\frac{1}{2}l(r-2)\frac{q}{q-2}}\right)^{\frac{q-2}{q}} \left(\sum_{l \leq 0} 2^{\frac{1}{2}l(r-2)\frac{q}{2}} (2^l J_{2^l}(\mathbb{E}_k g(x) : k \in \mathbb{Z})^{1/2})^q\right)^{\frac{2}{q}} > 1\right\}.$$

Define

$$A_{q,r} = \left(\sum_{l \leq 0} 2^{\frac{1}{2}l(r-2)\frac{q}{q-2}}\right)^{\frac{q-2}{q}} = \mathcal{O}((r-2)^{-\frac{q-2}{q}}).$$

Now Theorem A.3 immediately leads to the majorization

$$\begin{aligned} & \left| \left\{ x \in G : V_r(\mathbb{E}_k f(x) : k \in \mathbb{Z}) > 1 \text{ and } Mf(x) < 1/2 \right\} \right| \\ & \leq \left| \left\{ x \in G : \sum_{l \leq 0} 2^{\frac{1}{2}l(r-2)\frac{q}{2}} (2^l J_{2^l}(\mathbb{E}_k g(x) : k \in \mathbb{Z})^{1/2})^q > A_{q,r}^{-\frac{q}{2}} \right\} \right| \\ & \lesssim A_{q,r}^{\frac{q}{2}} \sum_{l \leq 0} 2^{\frac{1}{2}l(r-2)\frac{q}{2}} \int_{\mathbb{R}^d} |g(x)|^q dx \lesssim (r-2)^{1-q/2-1} \int_{\mathbb{R}^d} |g(x)|^q dx \lesssim (r-2)^{-q/2} \int_{\mathbb{R}^d} |g(x)|^q dx. \end{aligned}$$

Next, the square function S is bounded from below on $L^q(\mathbb{R}^d)$, therefore

$$\int_{\mathbb{R}^d} |g(x)|^q dx \lesssim \int_{\mathbb{R}^d} Sg(x)^q dx = \int_{\mathbb{R}^d} \left(\sum_{k \in \mathbb{Z}} |\mathbb{D}_k f(x)|^2 \cdot \mathbb{1}_{U_{k-1}}(x) \right)^{q/2} dx.$$

Since for $x \in U_{k-1}$, we have $\mathbb{E}_{k-1} \mathbb{1}_B(x) \leq 1/(2R)$, by the doubling property (A.3) we get $\mathbb{E}_k \mathbb{1}_B(x) \leq 1/2$. Hence,

$$\mathbb{1}_{U_{k-1}}(x) \leq 2 \cdot \mathbb{E}_k \mathbb{1}_{B^c}(x),$$

and

$$\int_{\mathbb{R}^d} |g(x)|^q dx \lesssim \int_{\mathbb{R}^d} \left(\sum_{k \in \mathbb{Z}} |\mathbb{D}_k f(x)|^2 \cdot \mathbb{E}_k \mathbb{1}_{B^c}(x) \right)^{q/2} dx.$$

We observe that for $q = 2$ we have

$$\int_{\mathbb{R}^d} \sum_{k \in \mathbb{Z}} |\mathbb{D}_k f(x)|^2 \cdot \mathbb{E}_k \mathbb{1}_{B^c}(x) dx = \int_{B^c} Sf(x)^2 dx.$$

For $q > 2$ let $\tilde{q} = q/2 > 1$ and \tilde{q}' be its dual exponent. Then for $h \in L^{\tilde{q}'}(\mathbb{R}^d)$

$$\begin{aligned} \int_{\mathbb{R}^d} \left(\sum_{k \in \mathbb{Z}} |\mathbb{D}_k f(x)|^2 \cdot \mathbb{E}_k \mathbb{1}_{B^c}(x) \right) h(x) dx &= \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^d} |\mathbb{D}_k f(x)|^2 \cdot \mathbb{E}_k \mathbb{1}_{B^c}(x) h(x) dx \\ &= \sum_{k \in \mathbb{Z}} \int_{B^c} |\mathbb{D}_k f(x)|^2 \cdot \mathbb{E}_k h(x) dx. \end{aligned}$$

Therefore, by Hölder's inequality, we get

$$\int_{\mathbb{R}^d} \left(\sum_{k \in \mathbb{Z}} |\mathbb{D}_k f(x)|^2 \cdot \mathbb{E}_k \mathbb{1}_{B^c}(x) \right) h(x) dx \leq \left(\int_{B^c} Sf(x)^q dx \right)^{2/q} \|Mh\|_{\tilde{q}'},$$

Taking the supremum over all $h \in L^{\tilde{q}'}(\mathbb{R}^d)$ we conclude

$$\int_{\mathbb{R}^d} \left(\sum_{k \in \mathbb{Z}} |\mathbb{D}_k f(x)|^2 \cdot \mathbb{E}_k \mathbb{1}_{B^c}(x) \right)^{q/2} dx \lesssim \int_{B^c} Sf(x)^q dx,$$

which finishes the proof. \square

Now, using Theorem A.1 we prove Lépingle's inequality for the sequence $(\mathbb{E}_k f : k \in \mathbb{Z})$.

Theorem A.5 ([13]). *For each $p \in (1, \infty)$ there exists $C_p > 0$ such that for all $r \in (2, \infty)$ and $f \in L^p(\mathbb{R}^d)$ we have*

$$\|V_r(\mathbb{E}_k f : k \in \mathbb{Z})\|_{L^p} \leq C_p \frac{r}{r-2} \|f\|_{L^p},$$

Moreover, $V_r(\mathbb{E}_k f : k \in \mathbb{Z})$ is also weak type $(1, 1)$.

Proof. Given $p > 1$ we take $q = 2p > 2$. By (A.4), for $f \in L^p(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ we have

$$\begin{aligned} C_q \cdot \left| \left\{ x \in \mathbb{R}^d : V_r(\mathbb{E}_k f : k \in \mathbb{Z}) > \lambda \right\} \right| &\leq \left| \left\{ x \in \mathbb{R}^d : Sf(x) > \lambda \right\} \right| + \left| \left\{ x \in \mathbb{R}^d : Mf(x) > \lambda/2 \right\} \right| \\ &\quad + \lambda^{-q} (r-2)^{-q/2} \int_{\{Sf \leq \lambda\}} Sf(x)^q dx. \end{aligned}$$

Therefore, we get

$$\begin{aligned}
\|V_r(\mathbb{E}_k f : k \in \mathbb{Z})\|_{L^p}^p &= p \int_0^\infty \lambda^{p-1} |\{x \in \mathbb{R}^d : V_r(\mathbb{E}_k f(x) : k \in \mathbb{Z}) > \lambda\}| d\lambda \\
&\lesssim \|Sf\|_{L^p}^p + \|Mf\|_{L^p}^p + (r-2)^{-p} \int_0^\infty \lambda^{p-q-1} \int_{\{S(f) \leq \lambda\}} S f(x)^q dx d\lambda \\
&\lesssim \|f\|_{L^p}^p + (r-2)^{-p} \int S f(x)^q \int_{S f(x)}^\infty \lambda^{p-q-1} d\lambda dx \\
&\lesssim \|f\|_{L^p}^p + (r-2)^{-p} \|Sf\|_{L^p}^p \lesssim (1 + (r-2)^{-p}) \|f\|_{L^p}^p.
\end{aligned}$$

For $p = 1$ it suffices to apply the Calderón–Zygmund decomposition and the desired claim follows. \square

Long variational bounds for \mathcal{M}_t follows the same line as in [10]. For every $f \in L^p(\mathbb{R}^d)$ with $p \in (1, \infty)$ we obtain

$$(A.6) \quad \|V_r(\mathcal{M}_{2^n} f : n \in \mathbb{Z})\|_{L^p} \leq \|V_r(\mathbb{E}_n f : n \in \mathbb{Z})\|_{L^p} + \left\| \left(\sum_{n \in \mathbb{N}} |\mathcal{M}_{2^n} f - \mathbb{E}_n f|^2 \right)^{1/2} \right\|_{L^p}.$$

The first term in (A.6) is bounded by Theorem A.5, whereas the square function can be estimated as in [10, Proof of Theorem 1.1]. In the next theorem we consider long variational estimates for \mathcal{T}_t .

Theorem A.6. *For every $p \in (1, \infty)$ there is $C_p > 0$ such that for all $r \in (2, \infty)$ and $f \in L^p(\mathbb{R}^d)$*

$$\|V_r(\mathcal{T}_{2^n} f : n \in \mathbb{Z})\|_{L^p} \leq C_p \frac{r}{r-2} \|f\|_{L^p}.$$

Proof. Let $\Phi \in C^\infty(\mathbb{R}^d)$ with compact support and integral one. Then we have the following decomposition (see [6])

$$\begin{aligned}
\mathcal{T}_{2^n} f &= \Phi_{2^n} * (\mathcal{T}f - \sum_{j < n} \mu_{2^j} * f) + (\delta_0 - \Phi_{2^n}) * \sum_{j \geq n} \mu_{2^j} * f \\
(A.7) \quad &= \Phi_{2^n} * \mathcal{T}f - (\Phi_{2^n} * \sum_{j < n} \mu_{2^j}) * f + \sum_{j \geq 0} (\delta_0 - \Phi_{2^n}) * \mu_{2^{j+n}} * f,
\end{aligned}$$

where

$$\mu_{2^j} * f(x) = \int_{G_{2^{j+1}} \setminus G_{2^j}} f(x - \mathcal{Q}(y)) K(y) dy.$$

The variational estimates for the first term in (A.7) follows by [10, Theorem 1.1, Lemma 2.1]. For the second term we apply the Littlewood–Paley theory since for each $n \in \mathbb{N}$

$$\Phi_{2^n} * \sum_{j < n} \mu_{2^j}$$

is a Schwartz function with integral zero. Thus, by (2.3),

$$V_r(\Phi_{2^n} * \sum_{j < n} \mu_{2^j} * f : n \in \mathbb{Z}) \lesssim \left(\sum_{n \in \mathbb{Z}} |\Phi_{2^n} * \sum_{j < n} \mu_{2^j} * f|^2 \right)^{1/2}.$$

For the last term we have

$$\|V_r(\sum_{j \geq 0} (\delta_0 - \Phi_{2^n}) * \mu_{2^{j+n}} * f : n \in \mathbb{Z})\|_{L^p} \leq \sum_{j \geq 0} \left\| \left(\sum_{n \in \mathbb{Z}} |(\delta_0 - \Phi_{2^n}) * \mu_{2^{j+n}} * f|^2 \right)^{1/2} \right\|_{L^p}$$

and due to the Littlewood–Paley theory one can show that there is $C_p > 0$ and $\delta_p > 0$ such that

$$\left\| \left(\sum_{n \in \mathbb{Z}} |(\delta_0 - \Phi_{2^n}) * \mu_{2^{j+n}} * f|^2 \right)^{1/2} \right\|_{L^p} \leq C_p 2^{-\delta_p j} \|f\|_{L^p}.$$

This completes the proof of Theorem A.6. \square

A.2. Short variations for averaging operators. To deal with short variations we need a counterpart of Lemma 2.2.

Lemma A.2. *Let $u < v$ be real numbers and $a : [u, v] \rightarrow \mathbb{C}$ be a differentiable function. For any $h \in \mathbb{N}$ and the sequence $(s_j : 0 \leq j \leq h)$ with $s_j = u + h^{-1}(v - u)j$ we have for every $r \in [1, \infty)$*

$$(A.8) \quad V_r(a(t) : t \in [u, v]) \lesssim \left(\sum_{j=0}^h |a(s_j)|^r \right)^{1/r} + \left(\sum_{j=0}^{h-1} \left(\int_{s_j}^{s_{j+1}} |a'(t)| dt \right)^r \right)^{1/r}.$$

Moreover, if $p \geq r$ then

$$\begin{aligned} & \left(\sum_{j=0}^h |a(s_j)|^r \right)^{1/r} + \left(\sum_{j=0}^{h-1} \left(\int_{s_j}^{s_{j+1}} |a'(t)| dt \right)^r \right)^{1/r} \\ & \lesssim h^{1/r-1/p} \left(\sum_{j=0}^h |a(s_j)|^p \right)^{1/p} + h^{1/r-1} (v-u)^{1-1/p} \left(\int_u^v |a'(t)|^p dt \right)^{1/p}. \end{aligned}$$

Proof. Fix $h \in \mathbb{N}$ and consider the sequence $(s_j : 0 \leq j \leq h)$ such that $s_j = u + h^{-1}(v - u)j$. Then

$$\begin{aligned} V_r(a(t) : t \in [u, v]) & \lesssim \left(\sum_{j=0}^h |a(s_j)|^r \right)^{1/r} + \left(\sum_{j=0}^{h-1} V_1(a(t) : t \in [s_j, s_{j+1}])^r \right)^{1/r} \\ & \lesssim \left(\sum_{j=0}^h |a(s_j)|^r \right)^{1/r} + \left(\sum_{j=0}^{h-1} \left(\int_{s_j}^{s_{j+1}} |a'(t)| dt \right)^r \right)^{1/r}. \end{aligned}$$

If $p \geq r$, by Hölder's inequality, we get

$$\begin{aligned} & \left(\sum_{j=0}^h |a(s_j)|^r \right)^{1/r} + \left(\sum_{j=0}^{h-1} \left(\int_{s_j}^{s_{j+1}} |a'(t)| dt \right)^r \right)^{1/r} \leq h^{1/r-1/p} \left(\sum_{j=0}^h |a(s_j)|^p \right)^{1/p} \\ & \quad + h^{1/r-1/p} \left(\sum_{j=0}^{h-1} \left(\int_{s_j}^{s_{j+1}} |a'(t)|^p dt \right)^{1/p} \right)^{1/p}. \end{aligned}$$

For the second term we again use Hölder's inequality to obtain

$$\begin{aligned} h^{1/r-1/p} \left(\sum_{j=0}^{h-1} \left(\int_{s_j}^{s_{j+1}} |a'(t)| dt \right)^p \right)^{1/p} & \lesssim h^{1/r-1/p} \left(\sum_{j=0}^{h-1} (s_{j+1} - s_j)^{p(1-1/p)} \left(\int_{s_j}^{s_{j+1}} |a'(t)|^p dt \right) \right)^{1/p} \\ & \lesssim h^{1/r-1} (v-u)^{1-1/p} \left(\int_u^v |a'(t)|^p dt \right)^{1/p} \end{aligned}$$

where in the last estimate we have used $s_{j+1} - s_j = (v-u)/h$. This completes the proof of the lemma. \square

Now the task is to prove that for every $p \in (1, \infty)$ there are $C_p > 0$ such that for every $f \in L^p(\mathbb{R}^d)$ we have

$$(A.9) \quad \left\| \left(\sum_{n \in \mathbb{Z}} V_2((\mathcal{M}_t - \mathcal{M}_{2^n})f : t \in [2^n, 2^{n+1}])^2 \right)^{1/2} \right\|_{L^p} \leq C_p \|f\|_{L^p}.$$

We may assume that f is a Schwartz function. Let S_j be a Littlewood–Paley projection $\mathcal{F}(S_j g)(\xi) = \phi_j(\xi) \mathcal{F}g(\xi)$ associated with $(\phi_j : j \in \mathbb{Z})$ a smooth partition of unity of $\mathbb{R}^d \setminus \{0\}$ such that for each $j \in \mathbb{Z}$ we have $0 \leq \phi_j \leq 1$ and

$$\text{supp } \phi_j \subseteq \{\xi \in \mathbb{R}^d : 2^{-j-1} < |\xi| < 2^{-j+1}\}$$

and for $\xi \in \mathbb{R}^d \setminus \{0\}$

$$\sum_{j \in \mathbb{Z}} \phi_j(\xi) = 1.$$

We are going to prove that for every $p \in (1, \infty)$ there are $C_p > 0$ and $\delta_p > 0$ such that for every $j \in \mathbb{Z}$ we have

$$(A.10) \quad \left\| \left(\sum_{n \in \mathbb{Z}} V_2((\mathcal{M}_t - \mathcal{M}_{2^n})S_{j+n}f : t \in [2^n, 2^{n+1}])^2 \right)^{1/2} \right\|_{L^p} \leq C_p 2^{-\delta_p |j|} \|f\|_{L^p}.$$

Applying (A.8) with $h = 2^{\varepsilon|j|}$, we obtain that

$$\begin{aligned} & \left\| \left(\sum_{n \in \mathbb{Z}} V_2((\mathcal{M}_t - \mathcal{M}_{2^n})S_{j+n}f : t \in [2^n, 2^{n+1}))^2 \right)^{1/2} \right\|_{L^p} \\ & \lesssim \left\| \left(\sum_{n \in \mathbb{Z}} \sum_{l=0}^h |(\mathcal{M}_{s_l} - \mathcal{M}_{2^n})S_{j+n}f|^2 \right)^{1/2} \right\|_{L^p} \\ & \quad + \left\| \left(\sum_{n \in \mathbb{Z}} \sum_{l=0}^{h-1} \left(\int_{s_l}^{s_{l+1}} \left| \frac{d}{dt} \mathcal{M}_t S_{j+n}f \right| dt \right)^2 \right)^{1/2} \right\|_{L^p} = I_p^1 + I_p^2. \end{aligned}$$

A.2.1. *The estimates for I_p^1 .* First, using vector-valued estimates from [15, Theorem A.1] together with the Littlewood–Paley theory we get

$$\begin{aligned} (A.11) \quad I_p^1 & \lesssim 2^{\varepsilon|j|/2} \left\| \left(\sum_{n \in \mathbb{Z}} \sup_{t > 0} |\mathcal{M}_t S_{j+n}f|^2 \right)^{1/2} \right\|_{L^p} \\ & \lesssim 2^{\varepsilon|j|/2} \left\| \left(\sum_{n \in \mathbb{N}} |S_{j+n}f|^2 \right)^{1/2} \right\|_{L^p} \lesssim 2^{\varepsilon|j|/2} \|f\|_{L^p}. \end{aligned}$$

Next, we are going to refine the estimate (A.11) for $p = 2$. Let m_t be the multiplier associated with the operator \mathcal{M}_t . By van der Corput's lemma [21], for each $t \in [2^n, 2^{n+1})$ we have

$$|m_t(\xi) - m_{2^n}(\xi)| \lesssim \min \{1, |2^{nA}\xi|_\infty, |2^{nA}\xi|_\infty^{-1/d}\}.$$

Therefore, by Plancherel's theorem

$$\begin{aligned} (A.12) \quad I_2^1 & = \left(\sum_{l=0}^h \int_{\mathbb{R}^d} \sum_{n \in \mathbb{Z}} |(m_{s_l}(\xi) - m_{2^n}(\xi))\phi_{j+n}(\xi)\mathcal{F}f(\xi)|^2 d\xi \right)^{1/2} \\ & \lesssim 2^{-|j|/d + \varepsilon|j|/2} \left(\int_{\mathbb{R}^d} \sum_{n \in \mathbb{Z}} |\phi_{j+n}(\xi)\mathcal{F}f(\xi)|^2 d\xi \right)^{1/2} \lesssim 2^{-|j|/d + \varepsilon|j|/2} \|f\|_{L^2}. \end{aligned}$$

Interpolating (A.11) with (A.12) and choosing appropriate $\varepsilon < 2/d$ we get

$$I_p^1 \lesssim 2^{-\delta_p|j|} \|f\|_{L^p}.$$

A.2.2. *The estimates for I_p^2 .* Since G is an open bounded convex set containing the origin, with a help of the spherical coordinates we may write

$$\mathcal{M}_t g(x) = \frac{1}{t^k |G|} \int_{S^{k-1}} \int_0^{r(\omega)t} g(x - \mathcal{Q}(r\omega)) r^{k-1} dr d\sigma(\omega)$$

where S^{k-1} is a unit sphere in \mathbb{R}^k and σ is the surface measure on S^{k-1} . We observe that if g is a Schwartz function

$$\begin{aligned} (A.13) \quad \frac{d}{dt} \mathcal{M}_t g(x) & = -k \frac{1}{t^{k+1} |G|} \int_{S^{k-1}} \int_0^{r(\omega)t} g(x - \mathcal{Q}(r\omega)) r^{k-1} dr d\sigma(\omega) \\ & \quad + \frac{1}{t^k |G|} \int_{S^{k-1}} g(x - \mathcal{Q}(r(\omega)t\omega)) r(\omega)^k t^{k-1} d\sigma(\omega). \end{aligned}$$

Change of the order of integration and differentiation is permitted since g is bounded. Hence, if $t \in [s_l, s_{l+1})$ and $s_l, s_{l+1} \simeq 2^n$, by Tonnelli's theorem, we get

$$\begin{aligned} \sum_{l=0}^{h-1} \int_{s_l}^{s_{l+1}} \left| \frac{d}{dt} \mathcal{M}_t g(x) \right| dt & \lesssim \mathcal{M}_{2^{n+1}} |g|(x) + \frac{1}{2^{nk} |G|} \int_{2^n}^{2^{n+1}} \int_{S^{k-1}} |g(x - \mathcal{Q}(r(\omega)t\omega))| r(\omega)^k t^{k-1} d\sigma(\omega) dt \\ & \lesssim \mathcal{M}_{2^{n+1}} |g|(x). \end{aligned}$$

Therefore, we obtain

$$(A.14) \quad I_p^2 \lesssim \left\| \left(\sum_{n \in \mathbb{Z}} (\mathcal{M}_{2^{n+1}} |S_{j+n}f|)^2 \right)^{1/2} \right\|_{L^p} \lesssim \left\| \left(\sum_{n \in \mathbb{Z}} \sup_{t > 0} (\mathcal{M}_t |S_{j+n}f|)^2 \right)^{1/2} \right\|_{L^p} \lesssim \|f\|_{L^p},$$

where the last inequality follows by the same line of reasoning as (A.11).

Next, we refine the estimates of I_p^2 for $p = 2$. Let \tilde{m}_t be the multiplier associated with the operator $\frac{d}{dt}\mathcal{M}_t$. We have

$$(A.15) \quad \tilde{m}_t(\xi) = -\frac{k}{t^{k+1}|G|} \int_{G_t} e^{2\pi i \xi \cdot \mathcal{Q}(x)} dx + \frac{1}{t^k|G|} \int_{S^{k-1}} e^{2\pi i \xi \cdot \mathcal{Q}(r(\omega)t\omega)} r(\omega)^k t^{k-1} d\sigma(\omega).$$

Indeed, by (A.13), we have

$$\begin{aligned} \mathcal{F}\left(\frac{d}{dt}\mathcal{M}_t g\right)(\xi) &= -\frac{k}{t^{k+1}|G|} \int_{G_t} e^{2\pi i \xi \cdot \mathcal{Q}(x)} dx \mathcal{F}g(\xi) \\ &\quad + \frac{1}{t^k|G|} \int_{\mathbb{R}^d} e^{2\pi i \xi \cdot x} \int_{S^{k-1}} g(x - \mathcal{Q}(r(\omega)t\omega)) r(\omega)^k t^{k-1} d\sigma(\omega) dx. \end{aligned}$$

Again, for the second term we need to justify the change of integrations. Let

$$R = \sup \{|\mathcal{Q}(y)| : y \in G_t\}.$$

Then for all $|x| \geq 2R$ and $|y| \leq R$ we have

$$|x - y| \geq \frac{|x|}{2},$$

thus

$$(A.16) \quad |g(x - \mathcal{Q}(y))| \lesssim (1 + |x|)^{-2d}.$$

By Fubini's theorem we get the claim. Moreover, we see that for $t \simeq 2^n$ we obtain $|\tilde{m}_t(\xi)| \lesssim 2^{-n}$.

Now, using (A.15), by the Cauchy–Schwarz inequality and Plancherel's theorem we get

$$\begin{aligned} (A.17) \quad I_2^2 &\leq \left\| \left(\sum_{n \in \mathbb{Z}} \frac{2^n}{h} \int_{2^n}^{2^{n+1}} \left| \frac{d}{dt} \mathcal{M}_t S_{j+n} f \right|^2 dt \right)^{1/2} \right\|_{L^2} \\ &= \left(\sum_{n \in \mathbb{Z}} \frac{2^n}{h} \int_{2^n}^{2^{n+1}} \int_{\mathbb{R}^d} |\tilde{m}_t(\xi) \phi_{j+n}(\xi) \mathcal{F}f(\xi)|^2 d\xi dt \right)^{1/2} \\ &\lesssim 2^{-\varepsilon|j|/2} \left(\sum_{n \in \mathbb{Z}} \int_{\mathbb{R}^d} |\phi_{j+n}(\xi) \mathcal{F}f(\xi)|^2 d\xi \right)^{1/2} \lesssim 2^{-\varepsilon|j|/2} \|f\|_{L^2} \end{aligned}$$

for $0 < \varepsilon < 1/d$. Thus interpolation of (A.14) with (A.17) gives

$$I_p^2 \lesssim 2^{-\delta_p|j|} \|f\|_{L^p}$$

and the proof of (A.10) is completed.

A.3. Short variations for truncated singular integral operators. We are going to show that for any $p \in (1, \infty)$ there are $C_p > 0$ and $\delta_0 > 0$ such that for every $j \in \mathbb{Z}$ and $f \in L^p(\mathbb{R}^d)$ we have

$$(A.18) \quad \left\| \left(\sum_{n \in \mathbb{Z}} V_2((\mathcal{T}_t - \mathcal{T}_{2^n}) S_{j+n} f : t \in [2^n, 2^{n+1}])^2 \right)^{1/2} \right\|_{L^p} \leq C_p 2^{-\delta_p|j|} \|f\|_{L^p}.$$

We may assume that f is a Schwartz function. The proof of (A.18) follows the same line as the one for the averaging operator. By Lemma A.2, for $h = 2^{\varepsilon|j|}$, we obtain

$$\begin{aligned} \left\| \left(\sum_{n \in \mathbb{Z}} V_2((\mathcal{T}_t - \mathcal{T}_{2^n}) S_{j+n} f : t \in [2^n, 2^{n+1}])^2 \right)^{1/2} \right\|_{L^p} &\lesssim \left\| \left(\sum_{n \in \mathbb{Z}} \sum_{l=0}^h |(\mathcal{T}_{s_l} - \mathcal{T}_{2^n}) S_{j+n} f|^2 \right)^{1/2} \right\|_{L^p} \\ &\quad + \left\| \left(\sum_{n \in \mathbb{Z}} \sum_{l=0}^{h-1} \left(\int_{s_l}^{s_{l+1}} \left| \frac{d}{dt} (\mathcal{T}_t - \mathcal{T}_{2^n}) S_{j+n} f \right|^2 dt \right)^{1/2} \right) \right\|_{L^p} = I_p^1 + I_p^2. \end{aligned}$$

A.3.1. *The estimates for I_p^1 .* By the vector-valued estimates from [15, Theorem A.1] and the Littlewood–Paley theory we get

$$(A.19) \quad \begin{aligned} I_p^1 &\lesssim 2^{\varepsilon|j|/2} \left\| \left(\sum_{n \in \mathbb{Z}} \sup_{t>0} (\mathcal{M}_t |S_{j+n}f|)^2 \right)^{1/2} \right\|_{L^p} \\ &\lesssim 2^{\varepsilon|j|/2} \left\| \left(\sum_{n \in \mathbb{N}} |S_{j+n}f|^2 \right)^{1/2} \right\|_{L^p} \lesssim 2^{\varepsilon|j|/2} \|f\|_{L^p}. \end{aligned}$$

For $p = 2$ we get better estimate. Let $m_{2^n, t}$ be the multiplier associated with the operator $\mathcal{T}_t - \mathcal{T}_{2^n}$ for $t \in [2^n, 2^{n+1})$. By van der Corput's lemma [21] (or more precisely the method of proof of van der Corput lemma from [21]) we have

$$|m_{t, 2^n}(\xi)| \lesssim \min \{1, |2^{nA}\xi|_\infty, |2^{nA}\xi|_\infty^{-1/d}\}.$$

Therefore, by Plancherel's theorem

$$(A.20) \quad \begin{aligned} I_2^1 &= \left(\sum_{l=0}^h \int_{\mathbb{R}^d} \sum_{n \in \mathbb{Z}} |(m_{s_l, 2^n}(\xi) \phi_{j+n}(\xi) \mathcal{F}f(\xi)|^2 d\xi \right)^{1/2} \\ &\lesssim 2^{-|j|/d + \varepsilon|j|/2} \left(\int_{\mathbb{R}^d} \sum_{n \in \mathbb{Z}} |\phi_{j+n}(\xi) \mathcal{F}f(\xi)|^2 d\xi \right)^{1/2} \lesssim 2^{-|j|/d + \varepsilon|j|/2} \|f\|_{L^2}. \end{aligned}$$

Interpolating (A.19) with (A.20) and choosing appropriate $\varepsilon < 2/d$ we get

$$I_p^1 \lesssim 2^{-\delta_p|j|} \|f\|_{L^p}.$$

A.3.2. *The estimates for I_p^2 .* Since G is an open bounded convex set containing the origin, with a help of the spherical coordinates we may write

$$(\mathcal{T}_t - \mathcal{T}_{2^n})g(x) = \int_{S^{k-1}} \int_{r(\omega)2^n}^{r(\omega)t} g(x - \mathcal{Q}(r\omega)) K(r\omega) r^{k-1} dr d\sigma(\omega)$$

where S^{k-1} is a unit sphere in \mathbb{R}^k and σ is the surface measure on S^{k-1} . We observe that if g is a Schwartz function

$$(A.21) \quad \frac{d}{dt}(\mathcal{T}_t - \mathcal{T}_{2^n})g(x) = t^{k-1} \int_{S^{k-1}} g(x - \mathcal{Q}(r(\omega)t\omega)) K(r(\omega)t\omega) r(\omega)^k d\sigma(\omega).$$

Change of the order of integration and differentiation is permitted since g is bounded and the kernel K satisfies (A.1). Hence, if $t \in [s_l, s_{l+1})$ and $s_l, s_{l+1} \simeq 2^n$, by Tonnelli's theorem, we get

$$\sum_{l=0}^{h-1} \int_{s_l}^{s_{l+1}} \left| \frac{d}{dt}(\mathcal{T}_t - \mathcal{T}_{2^n})g(x) \right| dt \lesssim \mathcal{M}_{2^{n+1}}|g|(x).$$

Therefore, we obtain

$$(A.22) \quad I_p^2 \lesssim \left\| \left(\sum_{n \in \mathbb{Z}} (\mathcal{M}_{2^{n+1}} |S_{j+n}f|)^2 \right)^{1/2} \right\|_{L^p} \lesssim \left\| \left(\sum_{n \in \mathbb{Z}} \sup_{t>0} (\mathcal{M}_t |S_{j+n}f|)^2 \right)^{1/2} \right\|_{L^p} \lesssim \|f\|_{L^p}.$$

Now we refine the estimate of I_p^2 for $p = 2$. Let $\tilde{m}_{t, 2^n}$ be the multiplier associated with the operator $\frac{d}{dt}(\mathcal{T}_t - \mathcal{T}_{2^n})$. We have

$$(A.23) \quad \tilde{m}_{t, 2^n}(\xi) = t^{k-1} \int_{S^{k-1}} e^{2\pi i \xi \cdot \mathcal{Q}(r(\omega)t\omega)} r(\omega)^k K(r(\omega)t\omega) d\sigma(\omega).$$

Indeed, by (A.21), we have

$$\mathcal{F}\left(\frac{d}{dt}(\mathcal{T}_t - \mathcal{T}_{2^n})g\right)(\xi) = t^{k-1} \int_{\mathbb{R}^d} e^{2\pi i \xi \cdot x} \int_{S^{k-1}} g(x - \mathcal{Q}(r(\omega)t\omega)) r(\omega)^k K(r(\omega)t\omega) d\sigma(\omega) dx$$

and thanks to estimates (A.1) and (A.16) we may change the order of integrations. Note that if $t \simeq 2^n$ we obtain $|\tilde{m}_{t, 2^n}(\xi)| \lesssim 2^{-n}$.

Now, using (A.23), by the Cauchy–Schwarz inequality and Plancherel’s theorem we get

$$\begin{aligned}
 (A.24) \quad I_2^2 &\leq \left\| \left(\sum_{n \in \mathbb{Z}} \frac{2^n}{h} \int_{2^n}^{2^{n+1}} \left| \frac{d}{dt} (\mathcal{T}_t - \mathcal{T}_{2^n}) S_{j+n} f \right|^2 dt \right)^{1/2} \right\|_{L^2} \\
 &= \left(\sum_{n \in \mathbb{Z}} \frac{2^n}{h} \int_{2^n}^{2^{n+1}} \int_{\mathbb{R}^d} |\tilde{m}_{t,2^n}(\xi) \phi_{j+n}(\xi) \mathcal{F}f(\xi)|^2 d\xi dt \right)^{1/2} \\
 &\lesssim 2^{-\varepsilon|j|/2} \left(\sum_{n \in \mathbb{Z}} \int_{\mathbb{R}^d} |\phi_{j+n}(\xi) \mathcal{F}f(\xi)|^2 d\xi \right)^{1/2} \lesssim 2^{-\varepsilon|j|/2} \|f\|_{L^2}
 \end{aligned}$$

for $0 < \varepsilon < 1/d$. Thus interpolation of (A.22) with (A.24) gives

$$I_p^2 \lesssim 2^{-\delta_p|j|} \|f\|_{L^p}$$

and the proof of (A.18) is completed.

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